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MATHEMATICAL METHODS IN ELECTROMAGNETIC THEORY

Abrupt Discontinuities: The Reflection Operator is a Contraction

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ABSTRACT: New basic properties of the reflection operator are analyzed for the problem of wave diffraction by abrupt discontinuities. A generalized (operator) form of the power conservation statement has been used to evaluate the norm of this operator and investigate the localization and the structure of its spectrum. The results obtained are useful for justifying matrix models of the considered class of diffraction problems, as well as for developing new methods of electrodynamic analysis of waveguiding and periodic structures.

INTRODUCTION

Solutions of wave diffraction problems involve at least two mutually complementary aspects. The first, or quantitative one is associated with a search for satisfactory numerical approximations to the scattering operators, thus representing a distinctly applied interest. As a result, methods of construction and numerical implementation of mathematical models of wave diffraction are widely presented in numerous publications.

The other aspect concerns determination of fundamental properties of the scattering operators. This is of decisive importance for validating the employed mathematical model, suggesting a rigorous analytical proof of convergence of the approximations to the true solution, investigating the relative convergence effect, etc. An adequate mathematical formalism in this case is the theory of operators in the Hilbert and Krein spaces. This aspect of the solution procedure has attracted only rather recently the attention of researchers, and remains poorly illuminated in the scientific literature.

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In papers [1-4] the Riemann-Hilbert problem technique and the method of spectral operators were applied to systematically investigate the properties of reflection and transmission operators for electromagnetic wave diffraction in the presence of periodic structures and waveguide discontinuities. Symmetry relations and spectral characteristics of the scattering operators were analyzed. In particular, paper [4] generalized the result of Mittra and Lee [5] concerning localization of the point spectrum of the reflection operator $\mathbf{R}$ inside of a unit disk. As was shown in paper [6], by way of example of a specific diffraction problem, knowledge of spectral properties of the reflection operator is sufficient for justifying the correctness of the matrix model constructed with the use of the mode-matching technique.

The present paper is aimed at analyzing new basic properties of the matrix-form reflection operator which follow from the fundamental laws of electrodynamics and are common for a wide class of diffraction problems concerning waveguide modes or spatial harmonics. To that end, the following approach is used in the paper.

That the entire spectrum of the reflection operator lies strictly inside the unit disk allows reducing the investigation of the operator $\mathbf{R}$ to an analysis of its Cayley transform. This replacement is justifiable since the power conservation law takes an essentially simpler form in terms of the Cayley transform, and hence the basic properties of the operator can be established much easier. The principal result of the paper follows immediately from the modified form of the power conservation statement; namely, the Cayley transform of the reflection operator is an accretive operator, and hence $\|\mathbf{R}\| \le 1$. Note that the respective theorem is proved in this paper with the use of geometrical properties of the Hilbert space, however the result sought for can also be obtained by applying the Hahn-Banach theorem.

The developed method is discussed below for the case of scalar problems of mode diffraction by an abrupt junction of two waveguiding channels (specific examples of this geometry are presented, e.g. in papers [1-6]). However, it should be noted that the suggested technique is also directly applicable to analyzing complex microwave devices with an arbitrary number of ports.

OPERATOR FORM OF THE POWER CONSERVATION LAW

Let sources α and β of a time harmonic field be present each in one of the two waveguiding channels through which M and N modes can propagate, respectively. The excitation of an arbitrary electromagnetic field is described by infinite-dimensional vectors ${}^{\alpha}\mathbf{b}$ and ${}^{\beta}\mathbf{b}$ with complex-valued components which have the physical meaning of specified amplitudes of a complete set of modes scattered by the abrupt discontinuity under analysis. These amplitudes will be normalized so as to meet the condition of a bounded energy in the incident field in the form ${}^{\alpha}\mathbf{b},\ {}^{\beta}\mathbf{b} \in \ell_2$.

Let us introduce the projection operator onto the propagating waves,

$$\mathbf{P}_K \equiv \left\{ p_{mn}^{(K)} = \sum_{q}^{K} \delta_{mq}\delta_{qn} \right\} = diag\left\{ \underbrace{1,...,1}_{K}, \underbrace{0,...}_{\infty} \right\}, \quad K = M\left(N\right), \tag{1}$$

which will select amplitudes of all the propagating modes in the form of the vectors ${}^{\alpha(\beta)}\mathbf{b}_{-} \equiv \mathbf{P}_K\, {}^{\alpha(\beta)}\mathbf{b}$. These form a $K-$ dimensional subspace $h_{-} \equiv \left\{ {}^{\alpha(\beta)}\mathbf{b}_{-} \right\} \subset \ell_2$ (δ_{mn} in Eq. (1) stands for the Kronecker delta symbol).

The remaining part of the field excitation vector, ${}^{\alpha(\beta)}\mathbf{b}_{+} = {}^{\alpha(\beta)}\mathbf{b} - {}^{\alpha(\beta)}\mathbf{b}_{-} = \mathbf{Q}_K\, {}^{\alpha(\beta)}\mathbf{b}$, with $\mathbf{Q}_K = \mathbf{I} - \mathbf{P}_K$, evidently, involves amplitudes of evanescent modes. All these vectors form an infinite-dimensional subspace $h_{+} \equiv \left\{ {}^{\alpha(\beta)}\mathbf{b}_{+} \right\} \subset \ell_2$. Such natural (from the physical standpoint) splitting of the initial Hilbert space gives rise to the Pontrjagin space, $\Pi_K \equiv h_{-} \cup h_{+},\ h_{-} \cap h_{+} = \varnothing$, characterized by the canonical symmetry, $\mathbf{J}_K = \mathbf{Q}_K - \mathbf{P}_K, \quad K = M\left(N\right)$.

In the further analysis a cardinal role belongs to the unitary operator which is defined by the equality

$$\mathbf{U}_K = \mathbf{Q}_K - i\,\mathbf{P}_K, \quad K = M\left(N\right) \tag{2}$$

and is closely related to the canonical symmetry according to the formula $\mathbf{U}_K^2 = \mathbf{U}_K^{-2} = \mathbf{J}_K$. As follows directly from the definition Eq. (2), the numerical range of this operator lies completely within the fourth quadrant of the complex plane. Using the terminology of paper [7], this operator is both accretive and accumulative simultaneously (in what follows, we will use the term “accretive-accumulative operator”).

As was shown in papers [6,8], by applying the matrix form of the reflection, ${}^{\alpha(\beta)}\mathbf{R} : \ell_2 \to \ell_2$, and transmission, ${}^{\alpha(\beta)}\mathbf{T} : \ell_2 \to \ell_2$, operators the power conservation law for the considered class of discontinuities can be brought to the following generalized form,

$$\left(\mathbf{I} + {}^{\alpha(\beta)}\mathbf{R}\right)\mathbf{U}_{M(N)}\left(\mathbf{I} - {}^{\alpha(\beta)}\mathbf{R}^{\dagger}\right) = {}^{\alpha(\beta)}\mathbf{T}\mathbf{U}_{N(M)}\, {}^{\alpha(\beta)}\mathbf{T}^{\dagger}, \tag{3}$$

where the dagger “†” denotes Hermitian conjugation. Note that application of the commonly known Lorentz lemma to the reciprocal waveguiding structure under analysis yields properties of the transposed matrix operators as follows ${}^{\alpha(\beta)}\mathbf{R}^T = {}^{\alpha(\beta)}\mathbf{R}$, ${}^{\alpha(\beta)}\mathbf{T}^T = {}^{\beta(\alpha)}\mathbf{T}$ (see, for example, papers [1,2]).

For further consideration it is sufficient to restrict ourselves to analyzing the scattering operators associated only with a single source. For this reason in what follows we will omit the superscripts designating these operators.

SPECTRUM OF THE REFLECTION OPERATOR AND ITS CAYLEY TRANSFORM

The equality

$$\operatorname{Im}\mathbf{R} \equiv \frac{1}{2i}\left(\mathbf{R}-\mathbf{R}^{\dagger}\right) = \mathbf{P}_M - \mathbf{V}\mathbf{P}_M\mathbf{V}^{\dagger} - \frac{1}{2}\mathbf{T}\mathbf{P}_N\mathbf{T}^{\dagger}, \tag{4}$$

represents the operator form of the power conservation law different from Eq.(3). Its validity can be checked by straight-forwardly substituting the expression for the newly introduced operator

$$\mathbf{V} = \frac{1}{\sqrt{2}}\left(\mathbf{I} - \mathbf{R}\,\mathbf{J}_M\right)$$

and accounting for the fundamental properties of the canonical symmetry. Note that the relation Eq. (4) has the meaning of the measure of nonself-adjointness of the reflection operator. According to the definition Eq. (1) the orthogonal projector $\mathbf{P}_{M(N)}$ represents an operator of finite rank. Hence, such is also the operator $\operatorname{Im}\mathbf{R}$ which follows from Eq. (4). Thus, the reflection operator is a nonself-adjoint one with a compact imaginary part (i.e, a quasi-Hermitian operator [9]) for the entire class of the diffraction problems under consideration.

Proceeding from this fact we can review some known results as for spectral properties of the reflection operator [4,5].

Theorem 1. The spectrum $\sigma(\mathbf{R})$ of the reflection operator lies completely inside a unit disk, with every non-real point of the spectrum being an eigenvalue of finite multiplicity, while the remaining parts of the spectrum may belong to the real axis only.

Proof. The properties of individual structural parts of the spectrum follows directly from the general theory of quasi-Hermitian operators [10]. Furthemore, let a row-vector $\mathbf{b}$, $\|\mathbf{b}\| = 1$, be known for any given $\varepsilon > 0$, such that the inequality $\|\mathbf{b}(\mathbf{R}-\lambda)\| \le \varepsilon$ holds (in the case of eigenvectors of the reflection operator we set $\varepsilon = 0$ and replace the inequality by an equality). By composing the left- and right-hand products of Eq. (3) with this vector, we will bring the left-hand part of the obtained equality to the form

$$\mathbf{b}\left[\left(\mathbf{I}+\lambda\right)+\left(\mathbf{R}-\lambda\right)\right]\mathbf{U}_M\left[\left(\mathbf{I}-\lambda^*\right)-\left(\mathbf{R}^\dagger-\lambda^*\right)\right]\mathbf{b}^\dagger=$$

$$=\left(1+\lambda\right)\left(1-\lambda^*\right)\mathbf{b}\mathbf{U}_M\mathbf{b}^\dagger+\mathrm{O}\left(\varepsilon\right).$$

By multiplying the total result by the number $\left(\mathbf{b}\mathbf{U}_M\mathbf{b}^\dagger\right)^\dagger=\mathbf{b}\mathbf{U}_M^\dagger\mathbf{b}^\dagger$, we get

$$\left(1+\lambda\right)\left(1-\lambda^*\right)\left|\mathbf{b}\mathbf{U}_M\mathbf{b}^\dagger\right|^2+\mathrm{O}\left(\varepsilon\right)=\left(\mathbf{b}\mathbf{T}\mathbf{U}_N\mathbf{T}^\dagger\mathbf{b}^\dagger\right)\left(\mathbf{b}\mathbf{U}_M^\dagger\mathbf{b}^\dagger\right).$$

Therefore, with $\varepsilon\to 0$ we have

$$sign\ \mathrm{Re}\left\{\left(1+\lambda\right)\left(1-\lambda^*\right)\right\}=sign\ \mathrm{Re}\left\{\left(\mathbf{b}\mathbf{T}\mathbf{U}_N\mathbf{T}^\dagger\mathbf{b}^\dagger\right)\left(\mathbf{b}\mathbf{U}_M^\dagger\mathbf{b}^\dagger\right)\right\}.$$

Taking into account the numerical range localization of the cramped unitary operators $\mathbf{U}_N$ and $\mathbf{U}_M^\dagger$, we arrive at the sought-for inequality $1-\left|\lambda\right|^2>0$.

Corollary 1. The reflection operator possesses a Cayley transform which in Weyl's notation takes the form

$$W\left(\mathbf{R}\right)\equiv\frac{\mathbf{I}+\mathbf{R}}{\mathbf{I}-\mathbf{R}}.$$

In virtue of the familiar properties of this transformation, the Cayley transform of the reflection operator, $\mathbf{D}=W\left(\mathbf{R}\right)$, is a quasi-Hermitian operator as well, and besides $\mathbf{D}^T=\mathbf{D}$.

Corollary 2. The spectrum, $\sigma\left(\mathbf{D}\right)$, of the Cayley transform lies entirely within the right-hand half-plane, with every non-real point of the spectrum being an eigenvalue of finite multiplicity, while the remaining parts of the spectrum may belong to the real axis only.

Proof. The spectrum mapping theorem,

$$\sigma\left(\mathbf{D}\right)=\sigma\left(W\left(\mathbf{R}\right)\right)=W\left(\sigma\left(\mathbf{R}\right)\right),\qquad(5)$$

suggests that if $\mu\in\sigma\left(\mathbf{D}\right)$, then $\mathrm{Re}\,\mu>0$. The structure of the spectrum $\sigma\left(\mathbf{D}\right)$ is determined both by the theorem Eq. (5) and the quasi-Hermiticity property of the Cayley transform.

CONTRACTION PROPERTY OF THE REFLECTION OPERATOR

Let us define the characteristic operator by the formula

$$\mathbf{G}_M \equiv \left(\mathbf{I}+\mathbf{R}\right)\mathbf{U}_M\left(\mathbf{I}-\mathbf{R}^{\dagger}\right). \tag{6}$$

Then the power conservation law Eq. (3) (in the form $\mathbf{G}_M = \mathbf{T}\mathbf{U}_N\mathbf{T}^{\dagger}$) implies that the introduced operator is an accretive-accumulative one.

Taking into account the spectral properties of the operator $\mathbf{D}$, let us represent the characteristic operator Eq. (6) through the Cayley transform, viz.

$$\frac{1}{4}\mathbf{G}_M = \left(\mathbf{D}+\mathbf{I}\right)^{-1}\mathbf{D}\mathbf{U}_M\left(\mathbf{D}^{\dagger}+\mathbf{I}\right)^{-1}.$$

As follows from this relation, the operator $\hat{\mathbf{D}} \equiv \mathbf{D}\mathbf{U}_M$ which is metrically equal to the Cayley transform occurs accretive-accumulative as well.

Now we are in possession of all the necessary data to prove the basic result of this paper. In the course of the proof two easily verified statements (see, for example, [11]) will be used.

Lemma. Let any number of the form $\lambda = -\alpha + i\beta$ with $\alpha > 0$ and $-\infty < \beta < \infty$ belong to the resolvent set of the operator $\mathbf{A}$ (i.e., $\lambda \in \rho\left(\mathbf{A}\right)$), then the inequality $\operatorname{Re}\mathbf{A} \equiv \frac{1}{2}\left(\mathbf{A}+\mathbf{A}^{\dagger}\right) \geq 0$ holds if and only if

$$\left\|\left(\mathbf{A}-\lambda\right)\mathbf{u}\right\| \geq \alpha\left\|\mathbf{u}\right\|, \quad \forall \mathbf{u} \in \ell_2$$

or, which is the same, $\left\|\left(\mathbf{A}-\lambda\right)^{-1}\right\| \leq \alpha^{-1}$.

Corollary. Let, for the same $\alpha > 0$ and $-\infty < \beta < \infty$, the number $-i\lambda = \beta + i\alpha$ belong to the set $\rho\left(\mathbf{B}\right)$, then $\operatorname{Im}\mathbf{B} \leq 0$ is valid if only if the inequality

$$\left\|\left(\mathbf{B}+i\lambda\right)\mathbf{u}\right\| \geq \alpha\left\|\mathbf{u}\right\|, \quad \forall \mathbf{u} \in \ell_2$$

holds or, which is equivalent, $\left\|\left(\mathbf{B}+i\lambda\right)^{-1}\right\| \leq \alpha^{-1}$.

The main result of this paper is the result of the following statement.

Theorem 2. The Cayley transform $\mathbf{D}$ is an accretive operator.

Proof. As was shown, the operator $\hat{\mathbf{D}}$ is an accretive and accumulative operator at a time. Then, according to the above lemma and the corollary thereof, the following estimates are valid,

$$\left.\begin{aligned} \left\|\left(\hat{\mathbf{D}}-\lambda\right)\mathbf{b}_{+}\right\| \geq \alpha\left\|\mathbf{b}_{+}\right\| \\ \left\|\left(\hat{\mathbf{D}}+i\lambda\right)\mathbf{b}_{-}\right\| \geq \alpha\left\|\mathbf{b}_{-}\right\| \end{aligned}\right\} \quad \forall \mathbf{b}_{\pm} \in \ell_2,$$

where $\lambda=-\alpha+i\beta\in\rho\left(\hat{\mathbf{D}}\right),\ \alpha>0$; $\mathbf{b}_{+}=\mathbf{Q}_M\mathbf{b}$, $\mathbf{b}_{-}=\mathbf{P}_M\mathbf{b}$. These two inequalities together yield

$$\left\|\left(\hat{\mathbf{D}}-\lambda\right)\mathbf{b}_{+}\right\|^2+\left\|\left(\hat{\mathbf{D}}+i\lambda\right)\mathbf{b}_{-}\right\|^2 \geq \alpha^2\left\|\mathbf{b}\right\|^2, \quad \forall \mathbf{b}\in\ell_2. \tag{7}$$

Let us transform the left part of (7) according to the parallelogram rule to the form

$$\left\|\left(\hat{\mathbf{D}}-\lambda\right)\mathbf{b}_{+}\right\|^2+\left\|\left(\hat{\mathbf{D}}+i\lambda\right)\mathbf{b}_{-}\right\|^2=\frac{1}{2}\left(\left\|\left(\mathbf{D}-\lambda\right)\mathbf{d}\right\|^2+\left\|\left(\mathbf{D}-\lambda\right)\mathbf{J}_M\mathbf{d}\right\|^2\right), \tag{8}$$

where the notation $\mathbf{d}=\mathbf{U}_M\mathbf{b}$ has been introduced.

Now, making use of the estimate

$$\left\|\left(\mathbf{D}-\lambda\right)\mathbf{d}\right\|^2+\left\|\left(\mathbf{D}-\lambda\right)\mathbf{J}_M\mathbf{d}\right\|^2 \leq 2\left\|\left(\mathbf{D}-\lambda\right)\mathbf{u}\right\|^2, \tag{9}$$

where $\left\|\left(\mathbf{D}-\lambda\right)\mathbf{u}\right\|=\max\left\{\left\|\left(\mathbf{D}-\lambda\right)\mathbf{d}\right\|,\left\|\left(\mathbf{D}-\lambda\right)\mathbf{J}_M\mathbf{d}\right\|\right\}$, for the right-hand part of Eq. (8), we can obtain from the two inequalities Eqs. (7) and (9), with account of the relation Eq. (8), the following result

$$\left\|\left(\mathbf{D}-\lambda\right)\mathbf{u}\right\|^2 \geq \alpha^2\left\|\mathbf{u}\right\|^2.$$

It has been taken into account here that $\left\|\mathbf{u}\right\|=\left\|\mathbf{d}\right\|=\left\|\mathbf{J}_M\mathbf{d}\right\|=\left\|\mathbf{b}\right\|$. Hence, we arrive at

$$\left\|\left(\mathbf{D}-\lambda\right)\mathbf{u}\right\| \geq \alpha\left\|\mathbf{u}\right\|, \quad \forall \mathbf{u}\in\ell_2.$$

Since $\lambda=-\alpha+i\beta\in\rho\left(\mathbf{D}\right),\ \alpha>0$, the repeated application of the lemma brings for the desired result, namely $\operatorname{Re}\mathbf{D}\geq 0$.

Corollary. The reflection operator is a contraction, $\|\mathbf{R}\| \le 1$, which is equivalent to that the operator $\mathbf{D}$ is accretive in virtue of the relation

$$\frac{1}{4}\begin{Bmatrix} \mathbf{I}-\mathbf{R}\mathbf{R}^{\dagger} \\ \mathbf{I}-\mathbf{R}^{\dagger}\mathbf{R} \end{Bmatrix} = \begin{Bmatrix} \left(\mathbf{D}+\mathbf{I}\right)^{-1} \\ \left(\mathbf{D}^{\dagger}+\mathbf{I}\right)^{-1} \end{Bmatrix} \mathrm{Re}\,\mathbf{D} \begin{Bmatrix} \left(\mathbf{D}^{\dagger}+\mathbf{I}\right)^{-1} \\ \left(\mathbf{D}+\mathbf{I}\right)^{-1} \end{Bmatrix}.$$

The latter is valid since $-1 \notin \sigma\left(\mathbf{D}\right), \sigma\left(\mathbf{D}^{\dagger}\right)$.

CONCLUSIONS

The power conservation law in its generalized form Eq. (3) determines fundamental properties of the operator of wave reflection from abrupt discontinuities in waveguiding structures (waveguide junctions, periodic structures with Floquet channels etc.). The method of analysis that has been proposed allows establishing the sought-for operator properties in the following sequence:

(1) the reflection operator $\mathbf{R}$ is a quasi-Hermitian one;

(2) the entire spectrum $\sigma\left(\mathbf{R}\right)$ lies strictly inside a unit disk, and, because of quasi-Hermitian nature of $\mathbf{R}$, all non-real points of the spectrum are eigenvalues of finite multiplicity;

(3) the operator $\mathbf{R}$ is a contraction.

It can be found in the same way that $\mathrm{Im}\left(\mathbf{R}\mathbf{J}_K\right), \mathrm{Im}\left(\mathbf{J}_K\mathbf{R}\right) \le 0$. To do that, note that the spectrum $\sigma\left(\mathbf{D}\mathbf{J}_K\right)$ lies within the lower half-plane, while the product $\mathbf{D}\mathbf{J}_K$ is an accumulative operator. Then it is necessary to use the interrelation of the operators $\mathrm{Im}\left(\mathbf{D}\mathbf{J}_K\right), \mathrm{Im}\left(\mathbf{J}_K\mathbf{D}\right)$ and $\mathrm{Im}\left(\mathbf{R}\mathbf{J}_K\right), \mathrm{Im}\left(\mathbf{J}_K\mathbf{R}\right)$, which follow from the Cayley transformation.

The established basic properties of the reflection operator and its Cayley transform will be useful for the rigorous justification of the computational electrodynamics methods based on modal analysis, and also for the development of new efficient algorithms of waveguide discontinuity analysis.

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