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A STUDY OF APPROXIMATION OF FUNCTIONS OF BOUNDED VARIATION BY FABER-SCHAUDER PARTIAL SUMS

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Система функцій Фабера-Шаудера була введена в 1910 році і стала першим прикладом базису в просторі функцій, неперервних на [0, 1]. Відомо низку результатів щодо властивостей рядів Фабера-Шаудера, у тому числі щодо оцінювання похибок наближення функцій поліномами та частинними сумами рядів, побудованих за системою Фабера-Шаудера. Відомо, що серед завдань теорії наближення функцій важливим є отримання нових оцінок величини наближення довільної функції деяким заданим класом функцій. Тому дослідження апроксимативних властивостей поліномів і частинних сум рядів Фабера-Шаудера становить значний інтерес для сучасної теорії апроксимації функцій.

Досліджено питання наближення функцій обмеженої варіації частинними сумами рядів, побудованих за системою функцій Фабера-Шаудера. Отримано оцінку похибки апроксимації функцій з класів функцій обмеженої варіації C_p ($1\leq p<\infty$) у метриці простору L_p за допомогою значень модуля неперервності дробового порядку $\varpi_{2-1/p}(f, t)$. З отриманої нерівності випливає оцінка похибки наближення неперервних функцій, яка виражена через модуль неперервності другого порядку.

Також у класі функцій C_p ($1<p<\infty$) отримані оцінки похибок наближення функцій у метриці простору L_p за допомогою модуля неперервності дробового порядку $\varpi_{1-1/p}(f, t)$.

Для класів функцій обмеженої варіації $KCV_{(2,p)}$ ($1\leq p<\infty$) отримано оцінку похибки наближення функцій у метриці простору L_p частинними сумами рядів Фабера-Шаудера.

Таким чином, отримано низку оцінок похибок наближення функцій обмеженої варіації їх частинними сумами рядів Фабера-Шаудера. Отримані результати є новими у теорії наближення функцій. Вони певним чином узагальнюють раніше відомі результати та можуть бути використані для подальших практичних застосувань.

Ключові слова: функції обмеженої варіації, інтегральна метрика, модуль неперервності, система Фабера-Шаудера.

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1. Introduction

The Faber-Schauder system of functions was introduced in the paper [1] and became the first example of a basis of the space of functions continuous on [0, 1]. Approximate properties of the Faber-Schauder system regarding approximation of individual functions and classes of functions are studied, for example, in [2–5]. In those studies, the upper bounds of errors of approximation of the function f continuous on [0, 1] by Faber-Schauder partial sums $\overline{S}_n(f,x)$ in the uniform metric are obtained.

Particularly, in [2] an estimate of the error of approximation of a continuous function by its Faber-Schauder partial sum is obtained. This result is specified in [4] using the second-order modulus of continuity.

In [6–8], the exact estimates of errors of approximation of functions from some function classes by Faber-Schauder partial sums in uniform and integral metrics are obtained.

However, the problems of approximation of functions of bounded variation by their Faber-Schauder partial sums have been investigated in few papers. In particular, in [9] considering the approximation of functions f from the classes C_p $(1\le p<\infty)$ by polynomials in the Faber-Schauder system, several estimates of upper bounds are obtained using the modulus of continuity of fractional orders $\omega_{k-1/p}(f,\delta)$.

The fact that there are known only few results regarding approximation of functions of bounded variation by Faber-Schauder partial sums can be explained particularly by certain complexity with obtaining the approximation errors of the functions by their Faber-Schauder partial sums in classes of functions of bounded variations.

Thus, investigation of approximation of functions of bounded variation by their Faber-Schauder partial sums and obtaining new results are of current interest not only to the modern theory of approximation but also to the wavelet theory actively used in modern signal processing. It is also appropriate to use moduli of continuity of fractional orders $\omega_{k-1/p}(f,\delta)$ for obtaining estimates of errors of approximation of functions by series in the Faber-Schauder system.

2. Literature review and problem statement

Although the Faber-Schauder system of functions was introduced in 1910 [1], investigation of the properties of the system, including approximate properties, began only in the 1950s with [2, 3]. Thus, investigating [2] the approximate properties of the Faber-Schauder system for an arbitrary continuous function, an upper bound of the value $\overline{\varepsilon}_n(f)_C$, in terms of the second-order modulus of continuity is obtained. Later, in [4] that result is specified and the following estimate of the error of approximation of an arbitrary continuous function by its Faber-Schauder partial sum is obtained:

$$f(x)-\overline{S}_n(f,x)\le 4\,\varpi_2\left(f;\frac{1}{n}\right),\quad n\ge 1.$$

In [3], an estimate of the error of approximation of an arbitrary continuous function by its partial Faber-Schauder sum using the first-order modulus of continuity is obtained.

$$f(x)-\overline{S}_n(f,x)\le 4\varpi\left(f,\frac{1}{n}\right).$$

Subsequently, that result is specified in [5] and the validity of the following relation is shown:

$$f(x)-\overline{S}_n(f,x)\le \frac{3}{2}\varpi\left(f,\frac{1}{n}\right),\quad n=2,3,\ldots.$$

It should be noted that in [2–5] only the questions of approximation of continuous functions in uniform metrics are considered and the obtained estimates are not exact in the sense of the final character of the estimates.

The first exact estimates of the errors of approximation of functions by partial sums in the Faber-Schauder system are obtained in [6–8]. In [6], the estimates of the errors of approximation of differentiable functions by their partial Faber-Schauder sums on classes of functions C^1 and W^1H_ω are obtained in integral metrics $\phi(L)$. Moreover, the estimates obtained in [6] can't be improved in case of a convex upward modulus of continuity.

In [7], the following unimprovable estimate of the error of approximation of differentiable functions from class L_∞^2 by Faber-Schauder partial sums in the metric L_∞ is obtained:

$$\left\|f(x)-\overline{S}_n(f,x)\right\|_{L_\infty}\le\frac{1}{8(n')^2}\left\|f^{(2)}\right\|_{L_\infty}.$$

Further studies in this direction are continued in [8] where a number of exact estimates of errors of approximation of the classes of differentiable functions L_p^1 by Faber-Schauder partial sums in integral metrics L_p are obtained.

However, the questions of approximation of functions of bounded variation by either polynomials or partial sums of series in the Faber-Schauder system aren't considered in the foregoing papers.

Only the work [9] is known, where the problems of approximation of functions of bounded variation by Faber-Schauder polynomials are studied with obtaining a number of estimates of approximation errors. Particularly, in [9] an upper bound of the error of the best approximation of functions f of bounded variation from the class C_p $(1\le p<\infty)$ by polynomials in the Faber-Schauder system in the space metric L_p is obtained:

$$E_n(f)_p\le 2^{4-3/p}\,\omega_{1-1/p}\left(f,\frac{1}{n}\right).$$

However, the questions of approximation of functions by Faber-Schauder partial sums aren't addressed in [9].

It should be also noted that studying the approximate properties of the Faber-Schauder system, the moduli of continuity of fractional orders $\omega_{k-1/p}(f,\delta)$ are used only in [9]. This is despite the fact that in connection with problems of approximation theory, the moduli of continuity of fractional orders $\omega_{k-1/p}(f,\delta)$ were first studied in [19] and used in several papers, for instance, [19–22], devoted to investigation of some questions of approximation theory, particularly to approximation of functions of bounded p-variation.

Application of the Faber-Schauder system in the theory of nonlinear approximation of functions is considered in [10]. In particular, some issues of the behavior of a greedy algorithm in the Faber-Schauder system in the space of continuous functions are examined [10].

As an example of a piecewise linear wavelet system that has been actively studied and used in recent decades in signal processing, the study of properties of the Faber-Schauder system is of considerable interest for the modern theory of functions, the theory of signal processing and wavelet theory.

In [11, 12], the behavior of the coefficients of decomposition of a continuous function in the Faber-Schauder series is investigated. The questions of convergence of series in the Faber-Schauder system are studied in [13–16]. In [17, 18], some questions of decomposition of functions in the Faber-Schauder system of functions are considered.

Consequently, taking into account the abovementioned, the properties of the Faber-Schauder system require further rigorous research. In particular, studying the approximation properties of the Faber-Schauder system and obtaining new results on estimates of errors of approximation of functions by polynomials and partial sums in the Faber-Schauder system are of importance for further investigations.

Using the moduli of continuity of fractional orders $\omega_{k-1/p}(f,\delta)$ is also of significance for obtaining new results on estimation of approximation errors in case of the Faber-Schauder system.

3. The aim and objectives of the study

The aim of the study is to consider the issues of approximation of functions of bounded variation by their Faber-Schauder partial sums. The classes of functions of bounded variation $C_p\ (1\le p<\infty)$ and $KCV_{(2,p)}\ (1\le p<\infty)$ are chosen for the investigation. Modules of continuity of fractional orders $\omega_{k-1/p}(f,\delta)\ (k=1,2)$ are chosen as characteristics of smoothness of the functions. To achieve the aim of the study, the following objectives are set up:

– to obtain estimates of errors of approximation of functions from classes of functions of bounded variation $C_p\ (1\le p<\infty)$ in the space metric L_p using the values of the moduli of continuity of fractional orders $\varpi_{1-1/p}(f,t)$ and $\varpi_{2-1/p}(f,t)$;

– in the class of functions of bounded variation $KCV_{(2,p)}$ $(1\le p<\infty)$, to obtain an estimate of the error of approximation of functions by Faber-Schauder partial sums in the metric $L_p\ (1\le p<\infty)$ applying the modulus of continuity of fractional orders $\varpi_{1-1/p}(f,t)$.

4. Definitions and notations necessary for further presentation of the results

Let us recall the necessary notations and definitions in order to formulate the results of the research.

Let $C\equiv C([0,1])$ be the space of continuous on $[0,1]$ functions f with the norm $\|f\|_C\overset{df}{=}\max\{|f(x)|:x\in[0,1]\}$, and let L_p $(1\le p<\infty)$ be the space of measurable functions f on $[0,1]$ whose p-th power is summable and hence the norm:

$$\|f\|_{L_p}\overset{df}{=}\left\{\int_0^1|f(x)|^p\,dx\right\}^{1/p}$$

is finite.

Let the function f be defined on $[0,1]$ and $\Pi\overset{df}{=}\{t_i\}_{i=0}^{s}$, where $0=t_0<t_1<\ldots<t_{s-1}<t_s=1$, is its arbitrary partition. The set of partitions of this type will be denoted by $\Im$. According to [23], the value:

$$\Lambda_p(f,\Pi)\overset{df}{=}\left\{\sum_{i=0}^{s}\left|f(t_i)-f(t_{i-1})\right|^p\right\}^{1/p}\quad(1\le p<\infty)$$

is called the variational sum of the order p of the function f by partition Π. If for the function f the following value is finite:

$$V_p(f)\overset{df}{=}\sup\{\Lambda_p(f,\Pi):\Pi\in\Im\}\quad(1\le p<\infty),$$

we say that the function f has a bounded p-variation on $[0,1]$. Let $V_p\ (1\le p<\infty)$ be the class of the functions f defined on $[0,1]$, for which $V_p(f)<\infty$ [24]. In case $p=1$, V_1 is a usual class of functions of bounded variation. In [23], it is shown that the functions f from the class $V_p\ (1\le p<\infty)$ can have the points of discontinuity of the first kind only. Therefore, if $f\in V_p\ (1\le p<\infty)$, then $f\in L_q$ for all $(1\le q<\infty)$.

According to [25], we assume that the function f given on $[0,1]$ belongs to the class $C_p\ (1\le p<\infty)$ if for any $\varepsilon>0$ there is the number $\delta=\delta(\varepsilon)>0$ such that the inequality:

$$\left\{\sum_i\left|f(\beta_i)-f(\alpha_i)\right|^p\right\}^{1/p}<\varepsilon$$

holds for an arbitrary finite system of disjoint intervals such that:

$$\left\{\sum_i(\beta_i-\alpha_i)^p\right\}^{1/p}<\delta.$$

The class C_1 is a class of absolutely continuous on $[0,1]$ functions. From the results of [26], it follows that the inclusions $C_p\subset C_r$ and $C_p\subset V_p$ where $1\le p<r<\infty$ are valid. Therefore, the classes $C_p\ (1\le p<\infty)$ are considered to be a generalization of the class C_1, and the functions included in them are called p-continuous functions. The property of p-continuity is considered to be an intermediate property between the properties of continuity $(p=\infty)$ and absolute continuity $(p=1)$ [27].

The modulus of continuity of fractional order $1-1/p$ $(1<p<\infty)$ for the function $f(x)\in V_p$ is called the value:

$$\omega_{1-1/p}(f,\tau)=\sup\{\Lambda_p(f,\Pi):\Pi\in\Im,|\Pi|\le\tau\},\qquad(1)$$

where $|\Pi|=\max\{t_i-t_{i-1}:i=\overline{1,s}\}$ is the diameter of partition Π [23].

Using the characteristic (1), it is shown in [25] that the class $C_p\ (1<p<\infty)$ coincides with the class of functions $f(x)\in V_p$ for which $\omega_{1-1/p}(f,\tau)\to0$ for $\tau\to0$.

The modulus of continuity of fractional order $k-1/p$ $(1<p<\infty)$ for the function $f\in V_p$ is defined in the following way [19]:

$$\omega_{k-1/p}(f,\delta)=\sup\{\omega_{1-1/p}(\Delta_\lambda^{k-1}f(x),\lambda):|\lambda|\le\delta\},\ k\in N_*,\quad(2)$$

where

$$\Delta_\lambda^m f(x)=\sum_{i=0}^{m}(-1)^{m-i}\binom{m}{i}f(x+i\lambda),\ m\in N.$$

Let the finite everywhere on $[0,1]$ function f has bounded (m,p)-variation $(1\le p<\infty)$ [28], [29] if

$$V_{(m,p)}(f)\overset{df}{=}\left\{\sup_{\xi}\sum_{k=1}^{n}\left|\sum_{v=0}^{m}(-1)^v C_m^\mu f(x_{k-1})+v\frac{x_k-x_{k-1}}{m}\right|\right\}^{1/p}<\infty,$$

where the upper bound is taken on all possible partitions $0=x_0<x_1<\ldots<x_n=1$ of the interval $[0,1]$. We define the class of functions with bounded (m,p)-variation $V_{(m,p)}(f)$ by $V_{(m,p)}$.

Let $KCV_{(m,p)}$ be the class of continuous on $[0,1]$ functions $f\in V_{(m,p)}$, the (m,p)-variations of which do not exceed the given positive number K.

We would like to note that in case $m=1$, the class of functions $V_{(1,p)}$ matches with the class of functions of bounded

p-variation V_p. In case $p=1$, the class $V_{(m,1)}$ was considered in [30]. In case $m=1$ and $p=1$, the class $V_{(2,1)}$ is considered in [31].

On the unit segment $[0,1]$, we introduce the dyadic intervals:

$$\delta_n \equiv \delta_m^k = \left((k-1)/2^m, k/2^m\right)$$

for the arbitrary number $n=2^m+k$ with $m\in Z_+$ and $k=1,...,2^m$.

The Haar system of functions is defined on $[0,1]$ in the following way (see, for example, [32, 33]): $\chi_1(t)\equiv\chi_0^{(0)}(t)\equiv 1$, and for every $n=2^m+k$ with $m\in Z_+$ and $k=\overline{1,2^m}$:

$$\chi_n(t)\equiv\chi_m^{(k)}(t)=\begin{cases}2^{m/2}, & \text{if } t\in\delta_{m+1}^{2k-1},\\ -2^{m/2}, & \text{if } t\in\delta_{m+1}^{2k},\\ 0, & \text{if } t\notin\overline{\delta_m^k},\end{cases} \quad (3)$$

where $\overline{M}$ is the closure of the set M. At the jump points, the Haar functions are equal to half the sum of their left and right limits. At the endpoints of $[0,1]$, they are equal to their limiting values from within $[0,1]$.

Using the Haar system of functions (3), the system of functions $\{\psi_n\}_{n\in Z_+}$ is defined in [1] in the following way:

$$\psi_0(x)\equiv 1;\quad \psi_n(x)=\int_0^x\chi_n(t)\,\mathrm{d}t\quad (n\in N; 0\le x\le 1).$$

It is shown in [1] that every continuous function $f\in C$ can be represented by the series:

$$f(x)=\sum_{k=0}^{\infty}a_k(f)\psi_k(x), \quad (4)$$

that converges uniformly on $[0,1]$, where the coefficients $a_k(f)$ are given by the formulae:

$$a_0(f)\overset{df}{=}f(0),\quad a_k(f)\overset{df}{=}\int_0^1\chi_k(x)\,\mathrm{d}f(x)\quad (k\in N). \quad (5)$$

The integral in (5) is understood in the Lebesgue-Stieltjes sense. The result (4) is replicated in [34] using the system of functions $\{\tilde{\psi}_n\}_{n\in Z+}$ that differ from $\{\psi_n\}_{n\in Z+}$ by constant factors only. For the n-th partial sum $(n\in N)$, we write the expression (4) as:

$$\overline{S}_n(f,x)\overset{df}{=}\sum_{k=0}^{n}a_k(f)\psi_k(x)\quad (n\in N). \quad (6)$$

The sum (6) is called the Faber-Shauder partial sum of the function $f\in C$. We introduce the quantity:

$$\overline{\varepsilon}_n(f)_X\overset{df}{=}\left\|f-\overline{S}_n(f)\right\|_X,$$

that is called the error of approximation of the function f by its Faber-Schauder partial sum $\overline{S}_n(f)$ in the space metric X.

5. Results of the study of approximation of functions from the classes C_p $(1\le p<\infty)$

Let $N_*=N\setminus\{1\}$ and $h\overset{df}{=}2^{-(m+1)}$. We also introduce the notation:

$$n'\overset{df}{=}\begin{cases}2^m, & \text{if } n=2^m+k\left(m\in N; k=\overline{1,2^m-1}\right),\\ 2^{m+1}, & \text{if } n=2^{m+1}\left(m\in Z_+\right).\end{cases} \quad (7)$$

Theorem 1. For all numbers $n=2^m+k$ $(m\in Z_+, k=\overline{1,2^m})$ and for the arbitrary function $f\in C_p$ $(1\le p<\infty)$, we have:

$$\overline{\varepsilon}_n(f)_{L_p}\le\frac{1}{(n')^{1/p}}\varpi_{2-1/p}\left(f;\frac{1}{2n'}\right). \quad (8)$$

Proof. Let there given the arbitrary function $f\in C$. We consider the following function on some interval $[\alpha,\beta]\subset[0,1]$:

$$\gamma(f;t;\alpha,\beta)\overset{df}{=}f(t)-\left\{f(\alpha)+\frac{f(\beta)-f(\alpha)}{\beta-\alpha}(t-\alpha)\right\}. \quad (9)$$

We have:

$$\gamma(f;\alpha;\alpha,\beta)=\gamma(f;\beta;\alpha,\beta)=0.$$

Let $\gamma(f;t;\alpha,\beta)\ne 0$ on the whole $[\alpha,\beta]$. We define by $t_{\max}$ the point on the interval $[\alpha,\beta]$, in which the function $|\gamma(f;t;\alpha,\beta)|$ reaches its highest value. We will consider two cases when the point $t_{\max}$ lies in the first and second half of the interval $[\alpha,\beta]$.

Let $|\alpha-t_{\max}|\le|\beta-t_{\max}|$. Then the point $t'=2t_{\max}-\alpha$ belongs to $[\alpha,\beta]$. Using the definition (9) and the fact that $t'-\alpha=2(t_{\max}-\alpha)$, we have:

$$\begin{aligned}&|\gamma(f;t_{\max};\alpha,\beta)|\le|\gamma(f;t_{\max};\alpha,\beta)+\\ &+\{\gamma(f;t_{\max};\alpha,\beta)-\gamma(f;t';\alpha,\beta)\}|=\\ &=|2f(t_{\max})-f(\alpha)-f(t')|=|f(t_{\max})+f(t_{\max}-\alpha)-\\ &-f(t_{\max})-f(\alpha+(t_{\max}-\alpha))+f(\alpha).\end{aligned} \quad (10)$$

In case if $[\beta-t_{\max}]\le|\alpha-t_{\max}|$, the point $t''=2t_{\max}-\beta$ belongs to the interval $[\alpha,\beta]$. Then similarly to the written above, we have:

$$\begin{aligned}&|\gamma(f;t_{\max},\alpha,\beta)|\le|2\gamma(f;t_{\max};\alpha,\beta)-\gamma(f;t'';\alpha,\beta)|=\\ &=|2f(t_{\max})-f(\beta)-f(t'')|=\\ &=\left|\begin{matrix}f(\beta+(t_{\max}-\beta))-f(\beta)-\\ -f(t_{\max}+(t_{\max}-\beta))+f(t_{\max})\end{matrix}\right|.\end{aligned} \quad (11)$$

We introduce the following notation:

$$I(f;t_{\max};\alpha,\beta)=\max\left\{\begin{matrix}\left|\begin{matrix}f(t_{\max}+(t_{\max}-\alpha))-f(t_{\max})-\\ -f(\alpha+(t_{\max}-\alpha))+f(\alpha)\end{matrix}\right|,\\ \left|\begin{matrix}f(\beta+(t_{\max}-\beta))-f(\beta)-\\ -f(t_{\max}+(t_{\max}-\beta))+f(t_{\max})\end{matrix}\right|\end{matrix}\right\}.$$

It is known (see, for example, [3], [8]) that the partial sum $\overline{S}_n(f;x)$ defined in (6) for any $n\in N_*$ is linear on the closed intervals $\overline{\delta_{m+1}^i}$ $(i=\overline{1,2k})$ and $\overline{\delta_m^j}$ $(j=\overline{k+1,2^m})$, and interpolates the function $f\in C$ at the points of the set D_n given by

$$D_1=\{0\}\cup\{1\},$$

$$D_n\overset{\mathrm{df}}{=}\left\{\frac{j}{2^m}\right\}_{j=0}^{2^m}\cup\left\{\frac{2j-1}{2^{m+1}}\right\}_{j=1}^{k}\quad (n=2,3,\ldots).$$

Thus, for the arbitrary function $f\in C$, we have:

$$\bar{\varepsilon}_n^p(f)_{L_p}=\left\|f-\bar{S}_n(f)\right\|_{L_p}^p=$$
$$=\sum_{i=1}^{2k}\int_{\delta_{m+1}^i}|f(t)-\bar{S}_n(f,t)|^p\,\mathrm{d}t+\sum_{j=k+1}^{2^m}\int_{\delta_m^j}|f(t)-\bar{S}_n(f,t)|^p\,\mathrm{d}t.$$

Then using (9) from the above equality, we can write:

$$\bar{\varepsilon}_n^p(f)_{L_p}=\sum_{i=1}^{2k}\int_{\delta_{m+1}^i}|\gamma(f;t;(i-1)h,ih)|^p\,\mathrm{d}t+$$
$$+\sum_{j=k+1}^{2^m}\int_{\delta_m^j}|\gamma(f;t;(j-1)2h,j2h)|^p\,\mathrm{d}t. \qquad (12)$$

Using (10), (11) and definition of the function $I(f;t_{\max};\alpha,\beta)$, we have:

$$\begin{cases}\int_{\delta_{m+1}^i}|\gamma(f;t;(i-1)h,ih)|^p\,\mathrm{d}t\le\\ \quad\le h\mathrm{I}^p(f;t_{\max,i};(i-1)h,ih),\ i=\overline{1,2k},\\ \int_{\delta_m^j}|\gamma(f;t;(j-1)2h,j2h)|^p\,\mathrm{d}t\le\\ \quad\le 2h\mathrm{I}^p(f;t_{\max,j};(j-1)2h,j2h),\ j=\overline{k+1,2^m},\end{cases} \qquad (13)$$

where $t_{\max,i}$ and $t_{\max,j}$ are the points in which the functions $|\gamma(f;t;(i-1)h,ih)|$ $(i=\overline{1,2k})$ and $|\gamma(f;t;(j-1)2h,j2h)|$ $(j=\overline{k+1,2^m})$ achieve their maximum values, respectively. From the definition (2), the modulus of continuity of the order 2–1/p can be written in the following form:

$$\varpi_{2-1/p}(f;\delta)=$$
$$=\sup\left\{\begin{matrix}\left[\sum_{i=1}^{n}|f(x_i+\lambda)-f(x_i)-f(x_{i-1}+\lambda)+f(x_{i-1})|^p\right]^{\frac{1}{p}}:\\ 0<|\lambda|\le\delta;\,x_i,x_i+\lambda\in[0,1],\,i=\overline{0,n}\end{matrix}\right\}. \qquad (14)$$

Then for any function $f\in C_p\,(1\le p<\infty)$ and $n=2^m+k$ with $m\in Z_+$ and $k=\overline{1,2^m-1}$ based on (10)–(13), definition $I(f;t_{\max};\alpha,\beta)$ and equality (14), we obtain from (12) the following:

$$\bar{\varepsilon}_n^p(f)_{L_p}\le 2h\left\{\begin{matrix}\frac{1}{2}\sum_{i=1}^{2k}I^p(f;t_{\max,i};(i-1)h,ih)+\\ +\sum_{j=k+1}^{2^m}I^p(f;t_{\max,j};(j-1)2h,2h)\end{matrix}\right\}\le$$
$$\le 2h\,\omega_{2-1/p}^p(f;h). \qquad (15)$$

If $n=2^{m+1}$, $m\in Z_+$, then using (9) for $f\in C$ we can write:

$$\bar{\varepsilon}_n^p(f)_{L_p}=\sum_{i=1}^{2^{m+1}}\int_{\delta_{m+1}^i}\left|f(t)-\bar{S}_n(f,t)\right|^p\mathrm{d}t=$$
$$=\sum_{i=1}^{2^{m+1}}\int_{\delta_{m+1}^i}\left|\gamma(f;t;(i-1)h,ih)\right|^p\mathrm{d}t. \qquad (16)$$

Then from (16), using (13)–(14) and taking into account the definition of the function $I(f;\ t_{\max};\ \alpha,\ \beta)$, for an arbitrary function $f\in C_p\,(1\le p<\infty)$ we have:

$$\bar{\varepsilon}_n^p(f)_{L_p}\le h\sum_{i=1}^{2k}I^p(f;t_{\max,i};(i-1)h,ih)\le h\varpi_{2-1/p}^p\left(f;\frac{h}{2}\right). \qquad (17)$$

We obtain the inequality (8) for any $n\in N_*$ and function $f\in C_p\ (1\le p<\infty)$ from (15), (17) and definition (7) of the numbers n'. Thus, Theorem 1 is proved.

It is known that in case $p\to\infty$, the space C_∞ coincides with the space of continuous functions C [19]. Then going to the limit $p\to\infty$ in (8), we can get the following result.

Corollary 1. For any $n\in N_*$ and function $f\in C$, we have:

$$\bar{\varepsilon}_n(f)_C\le\varpi_2\left(f;\frac{1}{2n'}\right), \qquad (18)$$

where $\varpi_2(f;x)$ is the second-order modulus of continuity [8]. This inequality is unimprovable on the set C.

The fact that inequality (18) cannot be improved on the set C can be proven applying the functions defined in [5].

The estimate (18) specifies one result obtained in [4].

Theorem 2. Let $1<p<\infty$ and $n=2^m+k$ with $m\in Z_+$, $k=\overline{1,2^m}$. Then for the arbitrary function $f(x)\in C_p$, the following inequality holds:

$$\bar{\varepsilon}_n(f)_{L_p}\le\frac{1}{(n')^{1/p}}\varpi_{1-1/p}\left(f;\frac{1}{n'}\right).$$

Proof. Let $f\in C$. Considering the function $\gamma(f;\ t;\ \alpha,\ \beta)$, defined in (9) on the fixed interval $[\alpha,\beta]\subset[0,1]$, we note that the value of the following quantity:

$$\left[f(\alpha)+\frac{f(\beta)-f(\alpha)}{\beta-\alpha}(t-\alpha)\right]$$

belongs to the interval $\left[\min(f(\alpha),f(\beta)),\max(f(\alpha),f(\beta))\right]$. Then using the notations from the proof of the Theorem 1, we define the following function:

$$Z(f;\ t_{\max};\ \alpha,\ \beta)\overset{df}{=}$$
$$\overset{df}{=}\max\left\{\begin{matrix}\left|f(t_{\max})-f(\alpha)\right|,\\ \left|f(t_{\max})-f(\beta)\right|\end{matrix}\right\}\ge\left|\gamma(f;\ t_{\max};\ \alpha,\ \beta)\right|. \qquad (19)$$

For the arbitrary function $f(x)\in C_p$ and any $n=2^m+k$ $(m\in N,k=\overline{1,2^m-1})$ from (12) and the notations (19) above, we have the following inequality:

$$\bar{\varepsilon}_n^p(f)_{L_p}\le\sum_{i=1}^{2k}\int_{(i-1)h}^{ih}\left|\gamma(f;t_{\max,i},(i-1)h,ih)\right|^p\mathrm{d}t+$$
$$+\sum_{j=k+1}^{2^m}\int_{(j-1)2h}^{j2h}\left|\gamma(f;t_{\max,j},(j-1)2h,j2h)\right|^p\mathrm{d}t\le$$
$$\le 2h\left\{\begin{matrix}\frac{1}{2}\sum_{i=1}^{2k}Z^p(f;t_{\max,i};(i-1)h,ih)+\\ +\sum_{j=k+1}^{2^m}Z^p(f;t_{\max,j};(j-1)2h,2h)\end{matrix}\right\}\le 2h\,\varpi_{1-1/p}^p(f;2h). \qquad (20)$$

In case $n=2^{m+1}$ $(m\in Z_+)$ from (16) for $f\in C_p$ $(1<p<\infty)$ we obtain:

$$\bar{\varepsilon}_n^p(f)_{L_p} \le h\sum_{i=1}^{2k} Z^p\left(f;\ t_{\max,i};\ (i-1)h, ih\right) \le$$
$$\le h\varpi_{1-1/p}^p(f;h). \qquad (21)$$

From the inequalities (20), (21) and definition (7), the inequality follows:

$$\bar{\varepsilon}_n(f)_{L_p} \le \frac{1}{(n')^{1/p}}\varpi_{1-1/p}\left(f;\frac{1}{n'}\right)$$

for any function $f \in C_p$ $(1 \le p < \infty)$. Thus, Theorem 2 is proved.

Going to the limit $p\to\infty$, the next result follows from the Theorem 2.

Corollary 2. For any function $f \in C$ and numbers $n \in N_*$, the following inequality holds:

$$\bar{\varepsilon}_n(f)_C \le \varpi\left(f;\frac{1}{n'}\right).$$

The inequality is unimprovable on the set C.

6. Results of the study of approximation of functions from the classes $KCV_{(2,p)}$ $(1 \le p < \infty)$

Let us further consider the approximation of the functions from the classes $KCV_{(2,p)}$ $(1 \le p < \infty)$.

Theorem 3. If the function $f \in KCV_{(2,p)}$, then for any $1 \le p < \infty$ and $n \in N_*$ the following inequality holds:

$$\bar{\varepsilon}_n(f)_{L_p} \le \frac{K}{(n')^{1/p}}. \qquad (22)$$

Proof. Let $f \in KCV_{(2,p)}$ and $(1 \le p < \infty)$. Taking into account the definition of the function $I(f;\ t_{\max};\ \alpha,\ \beta)$ and definition of the $(2, p)$-variation for $n = 2^m + k$ with $m \in N$ and $k = \overline{1, 2^m - 1}$ from (12), (13) and using (10), (11), we obtain the following inequality:

$$\bar{\varepsilon}_n^p(f)_{L_p} \le 2h\,K^p. \qquad (23)$$

In case if $n = 2^{m+1}$ $(m \in Z_+)$, then using the notations above, from (13) and (16) we have:

$$\bar{\varepsilon}_n(f)_{L_p} \le hK^p. \qquad (24)$$

The inequality (22) follows from (23), (24) and the definition (7) of the numbers n'.

7. Discussion of the results on studying the approximation of functions of bounded variation by Faber-Schauder partial sums

The issues of approximation of functions from the classes of functions of bounded variation by their Faber-Schauder partial sums and obtaining the estimates of errors of approximation of functions are studied. In particular, the classes of functions of bounded variation C_p and $KCV_{(2,p)}$ $(1 \le p < \infty)$ are considered.

In order to obtain the estimates of approximation errors of functions from the classes C_p by their Faber-Schauder partial sums, the modulus of continuity of fractional orders $\omega_{k-1/p}(f,\delta)$ that were not previously used when studying the problems of approximation of functions by Faber-Schauder partial sums are used.

New results for the approximation theory that can be used for further practical applications are obtained. The obtained results are new and generalize in some way the results known from [4].

Although the issues of approximation of functions of bounded variation from the classes C_p and $KCV_{(2,p)}$ $(1 \le p < \infty)$ by Faber-Schauder partial sums are investigated, the obtained results can be further extended for the case of approximation of functions by polynomials in the Faber-Schauder system.

It is also important to further investigate the approximation of functions of both one and many variables from other classes of functions of bounded variation and obtain new estimates of the errors of approximation of functions by polynomials and partial sums in the Faber-Schauder system.

The results of the research complement the known approximation properties of the Faber-Schauder system and establish the preconditions for further research in this direction.

New results are obtained from the theory of function approximation, which can be used for further practical applications, in particular, wavelet theory.

An applied aspect of using the obtained scientific results is the possibility of applying estimates of approximation errors in the theory of numerical methods in the construction of numerical algorithms, as well as in signal processing.

8. Conclusions

1. In the metric space L_p, new estimates of errors of approximation of functions from the classes C_p $(1 \le p < \infty)$ by Faber-Schauder partial sums using the values of the moduli of continuity of fractional orders $\varpi_{1-1/p}(f,t)$ and $\varpi_{2-1/p}(f,t)$ are obtained. The obtained results generalize in a certain way the results obtained earlier in [4].

2. The estimate of the error of approximation of functions of bounded variation from the classes $KCV_{(2,p)}$ $(1 \le p < \infty)$ in the metric L_p is obtained using the modulus of continuity of fractional order $\varpi_{1-1/p}(f,t)$.

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