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MATHEMATICAL METHODS IN ELECTROMAGNETIC THEORY

GENERALIZED MODE-MATCHING TECHNIQUE IN THE THEORY OF GUIDED WAVE DIFFRACTION. PART 2: CONVERGENCE OF PROJECTION APPROXIMATIONS*

I.V. Petrusenko♦ *& Yu.K. Sirenko*

A. Usikov Institute of Radio Physics and Electronics,
National Academy of Sciences of Ukraine
12, Academician Proskura St., Kharkiv 61085, Ukraine

♦Address all correspondence to I.V. Petrusenko E-mail: petrusigor@yahoo.com

A rigorous justification of applicability of the truncation procedure to solution of infinite matrix equation of the mode-matching technique still remains an open question throughout the years of its intensive use. The generalized mode-matching technique suggested for solving the problems of mode diffraction by a step-like discontinuity in a waveguide leads to the Fresnel formulas for matrix operators of wave reflection and transmission, rather than to standard infinite systems of linear algebraic equations. The present paper is aimed at constructing projection approximations for the mentioned operator-based Fresnel formulas and investigating analytically the qualitative characteristics of their convergence. To that end the theory of operators in the Hilbert space is used. The unconditional strong convergence of the finite-dimensional approximations of the operator-based Fresnel formulas to the true scattering operators is proved analytically. The condition number of the truncated matrix equation is estimated. The obtained results can be used for a rigorous justification of the mode-matching technique intended for efficient analysis of microwave devices.

KEY WORDS: *mode-matching technique, truncation of matrix operator, operator-based Fresnel formulas, projection convergence*

1. INTRODUCTION

Substantiation of the truncation procedure to solution of matrix-operator equations of the conventional mode-matching technique relates to those important problems which

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remained formidable throughout the years of its widespread use in computational electromagnetics.

Usually the mode-matching technique goes beyond the frame of the well-developed theory of the so-called “projection methods” of solving the linear equation

$$L\,x = f\,, \quad x, f \in H\,, \tag{1}$$

with a bounded operator $L\colon H \to H$ which acts in the Hilbert space H (the theory of these methods is presented, for example, in references [1-3]). Let $H_n \subset H$, $n = 1, 2, \ldots$ be a sequence of finite-dimensional subspaces and P_n be a projection operator $H \to H_n$. According to the projection method the sought-for approximation is represented by the solution of the equation

$$P_n L\, x^{(n)} = P_n f\,, \quad x^{(n)} \in H_n \tag{2}$$

(the necessary and sufficient conditions of strong convergence $\left\| x - x^{(n)} \right\| \to 0,\ n \to 0$, are presented, for example, in references [2-4]). The usability condition of such approach, i.e., that the operator L should belong to the class of “proper operators” [4], is reduced to that the operator L should be two-sided invertible and the so-called “Polskiy’s condition” [2] were met. The latter is equivalent to the requirement of invertibility of the operator $P_n L\colon H_n \to H_n$ for all values of $n > n_0$.

2. CONVERGENCE OF PROJECTION APPROXIMATIONS

As is known, the standard version of the mode-matching technique always leads to the mathematical model Eq. (1) in the form of an infinite system of linear algebraic equations for which it is usually easy to prove the boundedness of the given matrix operator. However, rigorous substantiation of the invertibility of this operator based on the asymptotic features of the elements of the infinite matrix of this system of linear equations has proven to be an insurmountable problem.

The difference of the conventional mode-matching technique is that it suggests two sequential approximations of the matrix operator (i.e., it is a “*fully discrete*” method). Namely, first an approximate operator $\hat{L}_m : H \to H$ is introduced which should give a certain approximate representation to the initial matrix operator $\hat{L}_m \to L$ with $m \to \infty$ and only then the projection scheme Eq. (2) is applied to the obtained approximate equation. The result is the following equation

$$P_n \hat{L}_m x^{(m,n)} = P_n f_m\,, \;\; x^{(m,n)} \in H_n\,,\; m = 1, 2, \ldots \tag{3}$$

Here f_m denotes the respective approximation to the right-hand part of Eq. (1) such that $f_m \to f$ with $m \to \infty$. Eq. (3) reveals the problem of the relative (or conditional) convergence intrinsic to the mode-matching technique; namely, whether the double proceeding to the limit $m, n \to \infty$ will lead to a result different from the true solution?

As it has been found, the infinite systems of linear algebraic equations Eq. (1) are not inherent to the mode-matching technique but rather follow from the conventional special formulation of the diffraction problem. If we change the problem formulation, for example, such as it has been suggested in paper [5], then we arrive at equations with respect to the matrix operators of wave reflection **R** and transmission **T** rather than to the equation Eq. (1). The infinite matrix model of the mode-matching technique for a class of problems of mode diffraction by a step-like discontinuity in a waveguide can be represented in the form of operator-based Fresnel formulas, viz.

$$\mathbf{R} = \frac{\mathbf{D}_0\mathbf{D}_0^T - \mathbf{I}}{\mathbf{D}_0\mathbf{D}_0^T + \mathbf{I}}; \quad \mathbf{T} = \left(\mathbf{D}_0\mathbf{D}_0^T + \mathbf{I}\right)^{-1} 2\,\mathbf{D}_0. \tag{4}$$

Here the given operator $\mathbf{D}_0$ is determined by the geometry of the problem and is dependent on the working frequency [5].

The present paper is aimed at constructing projection approximations for the operator-based Fresnel formulas Eq. (4) and investigating analytically qualitative characteristics of their convergence.

The conventional way of implementing numerically the matrix-operator model of the mode-matching technique is the truncation procedure. According to the latter the given operator of the problem is changed by a finite $M \times M$ matrix whose elements are approximate representations of the initial matrix operator $\mathbf{D} \equiv \mathbf{D}_0\mathbf{D}_0^T$. For the diffraction problem under consideration each element of this matrix represents N-th partial sum of a series. Then, an approximation to the sought-for solution is determined numerically for certain finite sequences of the numbers M and N (the so-called "practical convergence"). In what follows, we will apply the "method of reduction" for determining finite-dimensional matrix approximations to the operator-based Fresnel formulas which, in contrast to the mentioned truncation technique, assumes the limit processing $M, N \to \infty$.

In this paper we will use the results and notation of paper [5] in which the developed approach is illustrated by way of example of the canonical scalar problem of diffraction of modes LM_{m0}, $m = 1, 2, ...$ and LE_{m1}, $m = 0, 1, ...$ by an abrupt jump of the transverse cross-section of a rectangular waveguide in the H- and E-plane, respectively.

By analogy with the formulas Eq. (10) of paper [5] let introduce the orthoprojectors

$$\mathbf{P}_K \equiv \left\{ P_{mn}^{(K)} = \sum_{p=(0)1}^{K} \delta_{mp}\delta_{pn} \right\}, \quad \mathbf{Q}_K \equiv \mathbf{I} - \mathbf{P}_K, \tag{5}$$

where $K = M, N$ means the number of waveguide modes taken into account in the partial regions under consideration. Next, suppose that the field in region p, $p = 1, 2$, has been reduced to a set of M modes whereas N modes are considered in the adjacent partial region.

According to the projection scheme Eq. (3) the sought-after finite-dimensional $M \times M$ matrix approximation to the reflection operator is as follows

$$\tilde{\mathbf{R}}_p = \frac{\tilde{\mathbf{D}}_p - \mathbf{P}_M}{\tilde{\mathbf{D}}_p + \mathbf{P}_M}, \ \tilde{\mathbf{D}}_p = \begin{cases} \mathbf{P}_M \mathbf{D}_0 \mathbf{P}_N \mathbf{D}_0^T \mathbf{P}_M, \ p = 1, \\ \mathbf{P}_M \mathbf{D}_0^T \mathbf{P}_N \mathbf{D}_0 \mathbf{P}_M, \ p = 2. \end{cases} \tag{6}$$

The formula Eq. (6) represents the Cayley transform (see, for example, [6]) and possesses the following basic properties

$$\begin{gathered} \tilde{\mathbf{R}}_p^T = \tilde{\mathbf{R}}_p \ \Leftrightarrow \ \tilde{\mathbf{D}}_p^T = \tilde{\mathbf{D}}_p; \ \left\| \tilde{\mathbf{R}}_p \right\| < 1 \ \Leftrightarrow \ \operatorname{Re} \tilde{\mathbf{D}}_p > 0; \\ \tilde{\mathbf{R}}_p = \mathbf{P}_M - 2\tilde{\mathbf{A}}_p, \ \tilde{\mathbf{A}}_p \equiv \left(\tilde{\mathbf{D}}_p + \mathbf{P}_M \right)^{-1} : \tilde{\mathbf{A}}_p^T = \tilde{\mathbf{A}}_p, \\ \operatorname{Re} \tilde{\mathbf{A}}_p > \tilde{\mathbf{A}}_p \tilde{\mathbf{A}}_p^\dagger, \ \left\| \tilde{\mathbf{A}}_p \right\| < 1. \end{gathered} \tag{7}$$

At that approximate representations for the mode transmission operators take the form ${}^{12}\tilde{\mathbf{T}} = 2\tilde{\mathbf{A}}_1 \mathbf{D}_0 \mathbf{P}_K$ and ${}^{21}\tilde{\mathbf{T}} = 2\tilde{\mathbf{A}}_2 \mathbf{D}_0^T \mathbf{P}_K$.

Note that the identity of the properties of the approximate representation Eq. (7) and those of the exact solution [5] is a corollary of the continuity condition for the energy flow through the discontinuity aperture. Indeed, demanding the approximate representations for the tangential field components for the two partial regions in the form of truncated expansions in waveguide modes to be equal on the reference plane we achieve the obedience of these approximations to the four energy laws [5]. At that the oscillating power theorem and the first Lorentz lemma lead to the following relations (in the notation of paper [5])

$$\begin{gathered} \begin{cases} {}^p\tilde{\mathbf{R}}^T = {}^p\tilde{\mathbf{R}}, \\ {}^{pq}\tilde{\mathbf{T}}^T = {}^{qp}\tilde{\mathbf{T}}, \end{cases} \quad \Rightarrow \quad \tilde{\mathbf{S}}^T = \tilde{\mathbf{S}}, \\ \begin{cases} {}^p\tilde{\mathbf{R}}^2 + {}^{pq}\tilde{\mathbf{T}}\,{}^{pq}\tilde{\mathbf{T}}^T = \tilde{\mathbf{I}}, \\ {}^p\tilde{\mathbf{R}}\,{}^{pq}\tilde{\mathbf{T}} + \left({}^q\tilde{\mathbf{R}}\,{}^{qp}\tilde{\mathbf{T}} \right)^T = 0, \end{cases} \Rightarrow \tilde{\mathbf{S}}^2 = \tilde{\mathbf{I}}. \end{gathered} \tag{8}$$

Here the $(M+N) \times (M+N)$ matrices $\tilde{\mathbf{S}}$ and $\tilde{\mathbf{I}}$ are the finite-dimensional approximation to the generalized scattering matrix and the unity matrix, respectively. In turn, the complex power theorem and the second Lorentz lemma yield the energy conservation law in the generalized form, viz.

$$\left(\tilde{\mathbf{I}}+\tilde{\mathbf{S}}\right)\tilde{\mathbf{U}}\left(\tilde{\mathbf{I}}-\tilde{\mathbf{S}}^{\dagger}\right)=0, \tag{9}$$

where

$$\tilde{\mathbf{U}}=\operatorname{diag}\left(\tilde{\mathbf{U}}_1,\tilde{\mathbf{U}}_2\right);\ \tilde{\mathbf{U}}_j\equiv\mathbf{P}_K\mathbf{U}_j\mathbf{P}_K;\ j=1,2;\ K=M,N$$

represents the finite-dimensional approximation to the portal operator defined earlier in the paper [5].

In terms of the Cayley transform Eq. (6) the energy conservation law

$$\left(\mathbf{P}_K+{}^{p}\tilde{\mathbf{R}}\right)\tilde{\mathbf{U}}_p\left(\mathbf{P}_K-{}^{p}\tilde{\mathbf{R}}^{\dagger}\right)={}^{pq}\tilde{\mathbf{T}}\,\tilde{\mathbf{U}}_q\,{}^{pq}\tilde{\mathbf{T}}^{\dagger} \tag{10}$$

takes the form

$$\left.\begin{matrix}\tilde{\mathbf{D}}_1\tilde{\mathbf{U}}_1\\ \tilde{\mathbf{U}}_1\tilde{\mathbf{D}}_1^{\dagger}\end{matrix}\right\}=\mathbf{P}_K\mathbf{D}_0\tilde{\mathbf{U}}_2\mathbf{D}_0^{\dagger}\mathbf{P}_K,\qquad \begin{pmatrix}LM\\ LE\end{pmatrix}$$
$$\left.\begin{matrix}\tilde{\mathbf{U}}_2\tilde{\mathbf{D}}_2^{\dagger}\\ \tilde{\mathbf{D}}_2\tilde{\mathbf{U}}_2\end{matrix}\right\}=\mathbf{P}_K\mathbf{D}_0^{T}\tilde{\mathbf{U}}_1\mathbf{D}_0^{*}\mathbf{P}_K. \tag{11}$$

Making use of the same reasoning as in the course of analyzing the exact solution, we find that $\operatorname{Re}\tilde{\mathbf{D}}_p>0,\ p=1,2$, for all magnitudes of the figures *M* and *N*. This means that the approximate solution Eq. (6) exists and is unique and hence, the above mentioned "Polsky's condition" is met for any number of the modes (both propagating and evanescent ones) taken into account in the two partial regions.

Making use of the properties of the exact and approximate scattering operators we obtain the following estimate for the developed projection approximations

$$\begin{aligned}&\frac{1}{2}\left\|\left(\mathbf{P}_M\mathbf{R}_p-\tilde{\mathbf{R}}_p\right)\mathbf{b}^T\right\|<\left\|\left(\mathbf{P}_M\mathbf{D}_p-\tilde{\mathbf{D}}_p\right)\mathbf{A}_p\mathbf{b}^T\right\|,\\ &\frac{1}{2}\left\|\left(\mathbf{P}_M\,{}^{pq}\mathbf{T}-{}^{pq}\tilde{\mathbf{T}}\right)\mathbf{b}^T\right\|<\left\|\left(\mathbf{P}_M\mathbf{D}_p-\tilde{\mathbf{D}}_p\right)\mathbf{A}_p\mathbf{d}^T\right\|,\\ &\mathbf{d}=\mathbf{b}\begin{cases}\mathbf{D}_0,\ p=1,\\ \mathbf{D}_0^T,\ p=2,\end{cases}\quad q\neq p,\ \ \forall\mathbf{b}\in\ell_2.\end{aligned} \tag{12}$$

These inequalities make it possible to consider the convergence in the form

$$\left.\begin{matrix}\left\|\left(\mathbf{P}_M\mathbf{R}_p-\tilde{\mathbf{R}}_p\right)\mathbf{b}^T\right\|\\ \left\|\left(\mathbf{P}_M\,{}^{pq}\mathbf{T}-{}^{pq}\tilde{\mathbf{T}}\right)\mathbf{b}^T\right\|\end{matrix}\right\}\to 0,\text{ with } M,N\to\infty,\forall\mathbf{b}\in\ell_2$$

known as the strong projection convergence (or *P*-convergence) [3]. So, according to the estimates Eq. (12) the proof of the strong *P*-convergence of the developed approximations is reduced to determining the condition of strong *P*-convergence of the known matrix $\tilde{\mathbf{D}}_p$ to the specified operator $\mathbf{D}_p$, $p = 1, 2$.

Theorem 3. The projection approximations $\tilde{\mathbf{R}}_p$ and ${}^{pq}\tilde{\mathbf{T}}$ always demonstrate a strong *P*-convergence to the corresponding scattering operators.

Proof. Making use of the formula Eq. (22) of paper [5] and definition Eq. (6) let us write the operator difference under investigation in the form

$$\mathbf{P}_M \mathbf{D}_1 - \tilde{\mathbf{D}}_1 = \mathbf{P}_M \mathbf{D}_0 \mathbf{Q}_N \mathbf{D}_0^T + \mathbf{P}_M \mathbf{D}_0 \mathbf{P}_N \mathbf{D}_0^T \mathbf{Q}_M \qquad (13)$$

(in the case $p = 2$ it is necessary here to make changes $\mathbf{D}_0 \to \mathbf{D}_0^T$ and $\mathbf{D}_0^T \to \mathbf{D}_0$). Assertion of the theorem follows directly from the representation Eq. (13) since in space ℓ_2 the orthoprojector $\mathbf{P}_K, K = M, N$, converges strongly (but non-uniformly) to the identity operator.

As a result, the conditional (or relative) strong *P*-convergence of the projection approximations is absent for the infinite matrix model of the generalized mode-matching technique in the form of the operator-based Fresnel formulas Eq. (4).

Finally, proceeding from the properties Eq. (7) we obtain a uniform estimate for the condition number of the given matrix, viz.

$$1 \le \operatorname{cond}\left(\tilde{\mathbf{A}}_p\right) \le 1 + \left\| \mathbf{D}_0 \right\| \left\| \mathbf{D}_0^T \right\| < \infty, \forall M, N, \qquad (14)$$

which secures the stability of the computations.

3. CONCLUSIONS

The formidable problem intrinsic to the conventional mode-matching technique concerning justification of the applicability of the reduction procedure to solution of the final infinite system of linear algebraic equations is a corollary of the commonly used special formulation of the diffraction problem rather than of the mode-matching technique as such.

The new formulation of the wave diffraction problem proposed in the paper [5] leads to the generalized mode-matching technique in which case the convergence of the projection approximations to the true solution of the problem can be rigorously substantiated in an analytical way.

By way of example of the operator-based Fresnel formulas, the projection approximations to the sought-for scattering operators have been developed for the canonical problem of wave diffraction by a step discontinuity in the *H*- and *E*-plane.

It has been proved rigorously that the complex power conservation law and the second Lorentz theorem in the operator form [7,8] provide the mandatory fulfillment of the so-called "Polskiy's condition" [2,4] for any number of modes considered in regular waveguides, which guarantees the convergence of the projection approximations Eq. (6). *Ipso facto* the applicability of the truncation procedure to the matrix-operator models of the generalized mode-matching techniques has been substantiated rigorously.

The strong *P*-convergence of the developed approximations to the true solution has been proved analytically. The absence of the relative (conditional) convergence of the approximations for the mathematical model of the generalized mode-matching technique in the form of the operator-based Fresnel formulas has been rigorously justified. It has been found that the condition number of the truncated matrix model Eq. (6) is a uniformly bounded value.

The developed and rigorously justified generalized mode-matching technique can be used for efficient solution of the problem of analyzing waveguide structures and microwave devices.

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