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MATHEMATICAL METHODS IN ELECTROMAGNETIC THEORY

FRESNEL FORMULAE FOR SCATTERING OPERATORS

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For the scalar problem of mode diffraction on the abrupt waveguide discontinuity the Fresnel formulae for the reflection and transmission matrix operators are derived using the mode-marching technique. This generalized form of the matrix model is an immediate corollary of the proposed new statement of the problem. Making use of the energy conservation law in operator form, the correctness of the obtained Fresnel formulae for the scattering operators is proved analytically. Thus, the developed approach makes it possible to substantiate completely the widely used mode-matching technique for the class of diffraction problems under consideration.

KEY WORDS*:* *mode-matching technique, mode diffraction, Cayley transform*

1. INTRODUCTION

The reflection of the time-harmonic plane wave from the infinitely thin plane boundary S of two different material media (i.e., from a step-like shift or an abrupt discontinuity of electromagnetic properties of space) is discussed thoroughly in the majority of manuals (here we restrict all possible references to the fundamental work [1]). An appropriate mathematical law found by A. Fresnel for transverse waves of an elastic ether follows straightforwardly from the boundary conditions

$$\begin{cases} \vec{E}_{1\tau}(\omega) = \vec{E}_{2\tau}(\omega); \\ \vec{H}_{1\tau}(\omega) = \vec{H}_{2\tau}(\omega) \end{cases} \quad \text{on the boundary } S, \tag{1}$$

also known as “the matching conditions” for tangential components of the electrical and magnetic phasors of the frequency ω. The Fresnel formulae take a particularly simple form under the normal incidence of the wave on the boundary S; for instance, they can be written as

$$R = (\pm)\frac{d^2 - 1}{d^2 + 1}; \quad T = \frac{2d}{d^2 + 1}; \quad d \equiv \sqrt{\theta_{21}} \begin{pmatrix} E_{\perp} \\ H_{\perp} \end{pmatrix} \tag{2}$$

for two possible $E_{\perp}$ and $H_{\perp}$ polarizations with respect to the observation plane. Here R is the reflection coefficient in the first medium and T denotes the factor of wave transmission from the first medium to the second one, whereas $\theta_{21} = \sqrt{\frac{\mu_2}{\varepsilon_2} \cdot \frac{\varepsilon_1}{\mu_1}}$ is the relative wave impedance/admittance. (Note the property of Eqs. (2), in the substitution $(\pm) \to (\mp)$, which provides for the formula for the reflection coefficient in the second medium).

It would appear reasonable that the law of the form as given in (2) will occur for all phenomena of wave diffraction when the matching conditions (1) are met and the Poynting vector of the incident wave is normal to the plane boundary of the step-wise variation in the waveguiding structure properties.

The goal of this paper is to substantiate rigorously this guess for the problem on N-furcation of a generalized waveguide, which simulates the class of scalar problems on mode diffraction by the abrupt waveguide discontinuity.

Relations (1) can also be regarded as the initial equalities of the mode-matching technique widely used in computational electromagnetics. Therefore in this paper we assert that for the considered class of wave diffraction problems the mathematical model of the mode-matching technique can be written in the form of the Fresnel formulae for the reflection and transmission matrix operators.

However, note that the commonly known version of the mode-matching technique (presented, for example, in the book [2]) results in the infinite systems of linear algebraic equations in which identifying the Fresnel formulae is quite a challenge. Therefore, at first we need to carry out the procedure of generalizing the mode-matching technique. In this qualitative modification of the method, the key point is that the unknown vector of the Fourier coefficients is replaced by the matrix scattering operator sought for. In the applied electrodynamics this constructive idea had appeared to be consistently realized for the first time in the work [3]. This approach will be subsequently referred to as the *matrix operator technique* [4].

The found Fresnel formulae in operator form lead to the far-reaching consequences. In this paper, the potentialities of the proposed approach will be demonstrated with the solution to the actual problem of rigorous substantiation of the matrix model for the mode diffraction on the abrupt discontinuity in the waveguide.

2. GEOMETRY OF THE PROBLEM AND THE MATCHING CONDITIONS FOR THE PHASORS

Consider the abrupt N-furcation of a generalized rectangular cross-section waveguide by the system of semi-infinite perfectly conducting finite-thickness screens. We will describe the geometry of the problem in the orthogonal coordinate system $\{\xi,\eta,\zeta\}$, which is a generalization of the 2-D Cartesian $\{x,z\}\equiv\{\eta,\zeta\}$ and polar $\{\rho,\varphi\}\equiv\{\eta,\zeta\}$ frames in the plane $\xi=0$. In this coordinate system, the considered regular waveguide is a homogeneous one along the Cartesian axis $O\xi$, whereas its perfectly conducting walls are formed by two coordinate surfaces $\eta=const_{1,2}$ separated by the interval Ω_1. Assume that the waves propagate along axis $O\zeta$, which is directed towards furcation.

An s-th screening layer that splits the regular waveguide is created by the coordinate surfaces of $\eta=const$ type; it has a thickness of Ω'_s, $s=\overline{1,N-1}$. An interval between the screening layers Ω_q, $q=\overline{2,N+1}$ is the width of q-th regular waveguide which is geometrically similar to the generalized waveguide under consideration. In the general case the second or/and $(N+1)$-th waveguide may be non existent; then we have an additional step discontinuity. A single reference plane is placed to the discontinuity plane

$$S=\{\eta\in\Omega,\ \zeta=0\};\quad \Omega\equiv\left(\bigcup_{q=2}^{N+1}\Omega_q\right)\bigcup\left(\bigcup_{s=1}^{N-1}\Omega'_s\right)=\Omega_1\,. \tag{3}$$

The time dependence $\exp(i\omega t)$ is omitted throughout.

Let us consider the diffraction of LM_{m0}-modes, $m=1,2,...,$ with ξ-components of the field $H_\xi=0, E_\xi\neq0$, or of LE_{n1}-modes, $n=0,1,...,$ with $E_\xi=0, H_\xi\neq0$, which corresponds to the H-plane and, accordingly, to the E-plane problem. In the q-th waveguide, $q=\overline{1,N+1}$, the travel of the m-th mode in the positive direction of axis $O\zeta$ is specified by the exponent $\exp\left(-{}^q\gamma_m\zeta\right)$, where ${}^q\gamma_m$ is the propagation constant such that $\operatorname{Re}{}^q\gamma_m=0,\ \operatorname{Im}{}^q\gamma_m>0$ or $\operatorname{Re}{}^q\gamma_m>0,\ \operatorname{Im}{}^q\gamma_m=0$. Next, let $\varphi_q(\eta)\equiv\left\{{}^q\varphi_m(\eta)\right\}_{m=(0)1}^{\infty}$ be the column-vector of the real-valued transverse eigenfunctions of the q-th waveguide. Their basic properties are briefly described as

$$\boldsymbol{\varphi}_q^T(\eta)\boldsymbol{\varphi}_q(\eta')=\delta(\eta-\eta');\quad \left(\boldsymbol{\varphi}_q,\boldsymbol{\varphi}_q^T\right)_{\Omega_q}=\mathbf{I}\,. \tag{4}$$

Here the Dirac delta has been used, the superscript T denotes the transposition; the corresponding parentheses symbolize the integration over the interval Ω_q and $\mathbf{I}$ is the identity matrix operator (the idem-factor).

Next, let the scalar function ${}^pU_q(\eta,\zeta)$ denote the sought-for phasor, which determines in the q-th waveguide all the components of the electromagnetic field whose source is in the p-th waveguide, $p,q=\overline{1,N+1}$. It would appear natural to assume that this source generates the field which is a *complete set* of modes $\{LM_{m0}\}_{m=1}^{\infty}$ or $\{LE_{n1}\}_{n=0}^{\infty}$ with *any prescribed distribution* of complex amplitudes that we will collect into the row-vector ${}^{p}\mathbf{b}=\left\{{}^{p}b_m\right\}_{m=(0)1}^{\infty}\in\ell_2$. Let us represent the south-for phasor in the form

$$ {}^{p}U_q(\eta,\zeta)={}^{p}\mathbf{b}\cdot{}^{p}\mathbf{u}_q(\eta,\zeta), \tag{5}$$

where each element of the column-vector ${}^{p}\mathbf{u}_q=\left\{{}^{p}u_m^{(q)}(\eta,\zeta)\right\}_{m=(0)1}^{\infty}$ is the solution to the respective boundary-value problem in the q-th regular waveguide. Namely, the function ${}^{p}u_m^{(q)}(\eta,\zeta)$ has to satisfy (a) the 2-D Helmholtz equation; (b) homogeneous boundary conditions on the surface of perfect conductors; (c) the condition at infinity for waveguides and (d) the condition of energy boundedness in any closed volume inside the domain of field determination. The validity of expression (5) stems obviously from the linearity of the boundary-value problem under consideration.

The field continuity conditions on the common boundary of the first and q-th, $q=\overline{2,N+1}$, waveguides have the form of equalities:

$$\begin{cases} {}^{p}U_1={}^{p}U_q; \\ \partial_\zeta {}^{p}U_1=\partial_\zeta {}^{p}U_q; \end{cases} \quad \eta\in\Omega_q;\ \zeta=0, \tag{6}$$

whereas on the remaining parts of reference plane (3) the homogeneous boundary conditions are satisfied:

$$\left.\begin{matrix} (LM-\text{modes})\ {}^{p}U_1=0; \\ (LE-\text{modes})\ \partial_\zeta {}^{p}U_1=0; \end{matrix}\right\} \ \eta\in\Omega'_s,\ \zeta=0;\ s=\overline{1,N-1}. \tag{7}$$

In equalities (6) and (7) the symbol $\partial_\zeta\equiv\partial/\partial\zeta$ stands for the differentiation operator. Substituting the formula (5) into Eqs. (6) we obtain the relations that hold true for *all* ${}^{p}\mathbf{b}\in\ell_2$. We then immediately arrive at the following equalities:

$$\begin{cases} {}^{p}\mathbf{u}_1={}^{p}\mathbf{u}_q; \\ \partial_\zeta {}^{p}\mathbf{u}_1=\partial_\zeta {}^{p}\mathbf{u}_q; \end{cases} \quad \eta\in\Omega_q,\ \zeta=0;\ \begin{matrix} p=\overline{1,N+1} \\ q=\overline{2,N+1}, \end{matrix} \tag{8}$$

which represent the sought-after matching conditions for phasors written as (5). Along similar lines, we deduce from (7) that the vector function ${}^{p}\mathbf{u}_1$ or its derivative $\partial_\zeta {}^{p}\mathbf{u}_1$

satisfies the homogeneous boundary conditions on the face ends of perfectly conducting screens.

3. FRESNEL FORMULAE IN OPERATOR FORM

In corresponding regular waveguides, the unknown function ${}^{p}\mathbf{u}_q\left(\eta,\zeta\right)$ can explicitly be expanded into a series according to orthonormal waveguide modes [4,5]. In this case, the vector nature of this function dictates the emergence of matrix scattering operators in these expansions.

Thus, we arrive at a new following formulation of mode-diffraction problem. The finite-power wave is scattered by a given discontinuity in the waveguide; the field of this wave is an infinite set of modes with any known distribution of amplitudes. It is necessary to find the matrix operators of mode reflection and transmission.

For the problem in question we write at once the required relations on the reference plane (3):

$$ {}^{p}\mathbf{u}_q\left(\eta,0\right)=\begin{cases}\left(\mathbf{I}+{}^{p}\mathbf{R}\right)\mathbf{I}_{\gamma p}^{-1/2}\varphi_p\left(\eta\right);\, q=p; & \\ {}^{pq}\mathbf{T}\,\mathbf{I}_{\gamma q}^{-1/2}\varphi_q\left(\eta\right); \quad q\neq p; & \eta\in\Omega_q; \\ 0, \quad \eta\in\Omega_s';\ s=\overline{1,N-1};\ (LM) & \end{cases} \tag{9} $$

$$ \left(\partial_\zeta {}^{p}\mathbf{u}_q\right)\left(\eta,0\right)=\begin{cases}\left(\mathbf{I}-{}^{p}\mathbf{R}\right)\mathbf{I}_{\gamma p}^{1/2}\ \varphi_p\left(\eta\right)\cdot\begin{Bmatrix}\left(-1\right); p=1\\ \left(+1\right); p\geq 2\end{Bmatrix}; q=p; & \\ -{}^{pq}\mathbf{T}\,\mathbf{I}_{\gamma q}^{1/2}\ \varphi_q\left(\eta\right)\cdot\begin{Bmatrix}\left(-1\right); q=1\\ \left(+1\right); q\geq 2\end{Bmatrix}; \quad q\neq p; & \eta\in\Omega_q; \\ 0, \quad \eta\in\Omega_s';\ s=\overline{1,N-1};\ (LE). & \end{cases} \tag{10} $$

Here we have introduced a reflection matrix operator in the p-th waveguide ${}^{p}\mathbf{R}:\ell_2\to\ell_2$ and a matrix operator of mode transmission from the p-th waveguide into the q-th waveguide ${}^{pq}\mathbf{T}:\ell_2\to\ell_2$. In formulae (9) and (10) the diagonal operator $\mathbf{I}_{\gamma p}^{\pm 1/2}=\left\{\delta_{mn}\left({}^{p}\gamma_m\right)^{\pm 1/2}\right\}_{n=(0)1}^{\infty}$ is defined under the condition that the cut-off points (for which ${}^{p}\gamma_m=0$ at certain m) are absent. Here δ_{mn} is the Kronecker delta.

Substituting (9) and (10) into the matching conditions (8), using the Galerkin procedure and taking into account the properties of eigenfunctions (4), we come to a formal solution of the problem in terms of scattering operators:

$$ {}^{p}\mathbf{R} = (\pm)\begin{cases} \dfrac{\mathbf{D}_1 - \mathbf{I}}{\mathbf{D}_1 + \mathbf{I}}; & p = 1; \\ \mathbf{I} - 2\tilde{\mathbf{D}}_{pp}; & p \geq 2, \end{cases} \begin{pmatrix} LM \\ LE \end{pmatrix} \quad {}^{p}\mathbf{R}^{T} = {}^{p}\mathbf{R}; \tag{11} $$

$$ {}^{pq}\mathbf{T} = \begin{cases} (\mathbf{D}_1 + \mathbf{I})^{-1} 2\, {}^{q}\mathbf{D}_0; & p = 1; \\ (\pm) 2\, \tilde{\mathbf{D}}_{pq};\ p, q \geq 2; \begin{pmatrix} LM \\ LE \end{pmatrix} \end{cases} \quad q \neq p; \quad {}^{pq}\mathbf{T}^{T} = {}^{qp}\mathbf{T}, \tag{12} $$

where

$$ {}^{q}\mathbf{D}_0 = \mathbf{I}_{\gamma 1}^{\pm 1/2} \left(\varphi_1, \varphi_q^{T}\right)_{\Omega_q} \mathbf{I}_{\gamma q}^{\mp 1/2}; \quad \begin{pmatrix} LM \\ LE \end{pmatrix} $$

$$ \mathbf{D}_1 = \sum_{q=2}^{N+1} {}^{q}\mathbf{D}_0 {}^{q}\mathbf{D}_0^{T}; \quad \tilde{\mathbf{D}}_{pq} = {}^{p}\mathbf{D}_0^{T} (\mathbf{D}_1 + \mathbf{I})^{-1}\, {}^{q}\mathbf{D}_0 . $$

An expression for the reflection operator in the p-th waveguide, $p \geq 2$, can also be recast in a more convenient form:

$$ {}^{p}\mathbf{R} = (\mp)\frac{\mathbf{D}_p - \mathbf{I}}{\mathbf{D}_p + \mathbf{I}}; \quad \begin{pmatrix} LM \\ LE \end{pmatrix}. \tag{13} $$

Here the new operator $\mathbf{D}_p$, $p \geq 2$, is connected to the operator $\hat{\mathbf{D}}_{pp}$ by the following relations

$$ \mathbf{D}_p = \tilde{\mathbf{D}}_{pp} \left(\mathbf{I} - \hat{\mathbf{D}}_{pp}\right)^{-1} \Leftrightarrow \tilde{\mathbf{D}}_{pp} = \mathbf{D}_p \left(\mathbf{D}_p + \mathbf{I}\right)^{-1}. $$

In the special case, $N = 1$, $\Omega_1 = \Omega_2 \cup \Omega'$, which is correspond to the canonical problem of mode diffraction on a step discontinuity in the waveguide, the solution (11) and (12) takes a rather simple form:

$$ \begin{cases} {}^{1}\mathbf{R} = (\pm)\dfrac{\mathbf{D}_0\mathbf{D}_0^{T} - \mathbf{I}}{\mathbf{D}_0\mathbf{D}_0^{T} + \mathbf{I}}; \\ {}^{12}\mathbf{T} = \left(\mathbf{D}_0\mathbf{D}_0^{T} + \mathbf{I}\right)^{-1} 2\mathbf{D}_0; \end{cases} \quad \mathbf{D}_0 = \mathbf{I}_{\gamma 1}^{\pm 1/2} \left(\varphi_1, \varphi_2^{T}\right)_{\Omega_2} \mathbf{I}_{\gamma 1}^{\mp 1/2}; \quad \begin{pmatrix} LM \\ LE \end{pmatrix}, \tag{14} $$

which is similar in form with the Fresnel formulae (2). In addition, note that in order to find the scattering operators in the 2-nd waveguide it would suffice to make substitutions $\mathbf{D}_0 \rightleftarrows \mathbf{D}_0^{T}$, $(\pm) \to (\mp)$ in formulae for scattering operators (14) (with the above-mentioned analogous property of Eqs. (2))

Thus, a new statement of the considered mode-diffraction problem leads to the solution in the form of the Fresnel formulae for scattering operators (11)-(14). As will be apparent from the next Section, the existence and boundedness of all afore-mentioned inverse operators is guaranteed by the power conservation law.

4. CORRECTNESS OF THE OPERATOR FRESNEL FORMULAE

We validate the correctness of the developed mathematical model by implying that there is some proof of existence and uniqueness of solution (11)-(14) as well as of its stability on a set of bounded matrix operators defined in the Hilbert space ℓ_2.

Note that for the finite value of $\theta_{21} \neq 0$ a pair of linear-fractional transformation formally follows from the first Fresnel formula (2):

$$R = \frac{\theta_{21} - 1}{\theta_{21} + 1} \Leftrightarrow \theta_{21} = \frac{1 + R}{1 - R}, \quad (15)$$

from which in its turn a two-sided implication follows:

$$\operatorname{Re}\theta_{21} > 0 \Leftrightarrow |R| < 1. \quad (16)$$

In electrodynamics terms, these inequalities are in agreement with the energy condition for ordinary passive media $(\varepsilon', \mu' > 0; -\infty < \varepsilon'', \mu'' \leq 0)$ and signify that the numerical Fresnel formulae (2), (15) are correct (i.e., $\theta_{21} \neq -1,\ R \neq 1$).

Let us show that the correctness of obtained operator Fresnel formulae (11)-(14) also follow from the fundamental energy law. Our proving is based on the previously established (see [4]) a certain duality of properties of the operators, which form the first Fresnel formula (11) or (13). Namely, if for the given matrix operator $\mathbf{D}_p$ the localization of its spectrum $\sigma(\mathbf{D}_p)$ is unknown, then it is just the basic characteristics of the entire spectrum $\sigma(\mathbf{R}_p)$ of the sought-for reflection operator $\mathbf{R}_p$ are completely defined by the generalized power conservation law [6]. The interrelation of these two operators in the form of the first Fresnel formula makes it possible to find all of their required properties.

Theorem 1. The Fresnel formulae (11)-(14) are the complete and consistent solution of the considered problem of mode diffraction on the abrupt discontinuity in the waveguide.

Proof. It amounts to substantiating the condition $-1 \notin \sigma(\mathbf{D}_p)$, which is equivalent to the existence of the bounded operator $(\mathbf{D}_p + 1)^{-1} : \ell_2 \to \ell_2,\ p = \overline{1, N+1}$.

The diagonal blocks of the generalized power conservation law (formula (20) from [6]) yield the following relation:

$$\left(\mathbf{I}+{}^{p}\mathbf{R}\right)\mathbf{U}_p\left(\mathbf{I}-{}^{p}\mathbf{R}^{\dagger}\right)=\sum_{q=1,q\neq p}^{N+1}{}^{pq}\mathbf{T}\mathbf{U}_q{}^{pq}\mathbf{T}^{\dagger},\quad p=\overline{1,N+1}, \tag{17}$$

where the dagger "†" is for Hermitian conjugation. It follows from the definition of the cramped unitary operator $\mathbf{U}_p=\left\{\delta_{mn}\exp\left[-i\arg\left({}^{p}\gamma_m\right)\right]\right\}_{m=(0)1}^{\infty}$ that its numerical range lies completely within the fourth quadrant of the complex plane. Hence, all the complex numbers of the type

$$\sum_{q=1,q\neq p}^{N+1}{}^{p}\mathbf{b}\;{}^{pq}\mathbf{T}\mathbf{U}_q{}^{pq}\mathbf{T}^{\dagger}\;{}^{p}\mathbf{b}^{\dagger},\quad\forall\,{}^{p}\mathbf{b}\in\ell_2$$

are also belong to the same quadrant. It then follows from (17) that the spectrum $\sigma\left({}^{p}\mathbf{R}\right)$ lies within the unit disc and each nonreal point of this spectrum is an eigenvalue of finite multiplicity [6,7] (see also [2,4,8]). Hence, there exists the Cayley transform, which in Weyl's notation takes the form [9]:

$$\mathbf{W}_p=\frac{\mathbf{I}+{}^{p}\mathbf{R}}{\mathbf{I}-{}^{p}\mathbf{R}}. \tag{18}$$

As corollary of the spectrum mapping theorem (see, e.g., [10]) the spectrum of the operator $\mathbf{W}_p$ lies entirely within the right-hand half-plane, $\mathrm{Re}\,\lambda>0,\ \forall\lambda\in\sigma\left(\mathbf{W}_p\right)$. The apparent relation

$$\mathbf{D}_p\equiv\begin{cases}\mathbf{W}_1^{\pm1},\ p=1;\\ \mathbf{W}_p^{\mp1},\ p\geq 2,\end{cases}\begin{pmatrix}LM\\ LE\end{pmatrix} \tag{19}$$

completes the proof.

Theorem 2. The operator $\mathbf{D}_p$ is an accretive one, whereas the reflection operator ${}^{p}\mathbf{R}$ is a contraction, $p=\overline{1,N+1}$.

Proof. The two-sided implication identical-in-form with Eq. (16)

$$\mathrm{Re}\,\mathbf{W}_p>0\ \Leftrightarrow\ \left\|{}^{p}\mathbf{R}\right\|<1$$

is the basic property of the Cayley transformation (18) (e.g. see [7]). Therefore it would suffice to prove the accretiveness of operator $\mathbf{D}_p$, i.e., $\mathrm{Re}\,\mathbf{D}_p>0$.

In terms of the Cayley transform the energy law (17) takes the form:

$$\mathbf{W}_p\mathbf{U}_p = \frac{1}{4}\sum_{q=1,q\neq p}^{N+1}\left[\left(\mathbf{W}_p + \mathbf{I}\right)^{pq}\mathbf{T}\right]\mathbf{U}_q\left[{}^{pq}\mathbf{T}^{\dagger}\left(\mathbf{W}_p^{\dagger} + \mathbf{I}\right)\right]. \tag{20}$$

From this equality it follows that the numerical range of operator $\mathbf{W}_p\mathbf{U}_p$ lies entirely within the fourth quadrant of the complex plane. The literally replicated proof of theorem 2 from the paper [7] results in equality $\operatorname{Re}\mathbf{W}_p > 0$. Identity (19) completes the proof.

The stability of the found solution (11)-(14) is established by the following

Theorem 3. The operator $\mathbf{A}_p = \left(\mathbf{I} + \mathbf{D}_p\right)^{-1}$ is an accretive contraction: $\operatorname{Re}\mathbf{A}_p > \mathbf{A}_p\mathbf{A}_p^{\dagger}$.

Proof. Through direct calculation we obtain

$$\operatorname{Re}\mathbf{A}_p > \mathbf{A}_p\mathbf{A}_p^{\dagger} + \mathbf{A}_p\left(\operatorname{Re}\mathbf{D}_p\right)\mathbf{A}_p^{\dagger} > 0$$

as a corollary to the accretiveness of operator $\mathbf{D}_p$.

5. CONCLUSIONS

The $H-\left(E-\right)$ plane $\left(N+1\right)$-port junction of the waveguides of rectangular cross-section, which are regular in the generalized frame above, has been analyzed via the mode-matching technique.

A new statement of the problem of mode diffraction on the discontinuity in the waveguide has been formulated. This formulation is as follows. The finite-power wave is incident upon the waveguide discontinuity. The field of this wave consists of an infinite set of modes with any known amplitude distribution. There are just the scattering operators to be found.

As a direct result of this statement of the mode-diffraction problem, the matrix-operator model in a perfect form of Fresnel formulae for the reflection and transmission operators has been obtained.

It has been found that the fundamental law of energy conservation implies the correctness of the derived Fresnel formulae in operator form. Thus, the unsettled problem of justification of the matrix model of the mode-matching technique has been solved for the considered class of diffraction problems.

Finally, owing to a new statement of the problem of mode diffraction the matrix-operator nature of the mode-matching technique has been clarified.

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