Electromagnetics, 24:237–254, 2004

ISSN: 0272-6343 print/1532-527X online
DOI: 10.1080/02726340490442599

Analytic-Numerical Analysis of Waveguide Bends

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A novel rigorous solution of the problem of mode diffraction by the junction between a straight and a uniformly curved rectangular waveguide is presented. The generalized scattering matrix of the unit is obtained via the matrix operator technique. The full-wave model is based on Cayley's transformation between a reflection and an accretive operator. The convergence of approximate solutions is established analytically. The computational efficiency of the method is demonstrated. The power conservation law and the reciprocity relations are used in the operator matrix form. The approach proposed can be useful for numerical-analytical solution of various electromagnetic problems.

Keywords waveguide bend, generalized mode-matching method, matrix operator technique, Cayley's transform

Introduction

Taking into account a great number of works published during more than six decades, mode diffraction by the uniform bend of a waveguide can be classified as a canonical problem. An excellent review and an extensive bibliography of earlier work on the circular bend of a waveguide were given by Cochran and Pecina (1966) and Lewin (1977). In addition, mention should be made of thorough theoretical and experimental investigations of wave propagation in a continuously curved guide carried out by Voskresenskii (1957). From the standpoint of the modern requirements for electrodynamic analysis, however, the earlier methods are of limited usefulness.

To date, an accurate and efficient model for the uniform bend of a waveguide has been a subject of constant interest to researchers who try to obtain an effective modal solution in the curved waveguide section. The point is that the modal (dispersion) equation found via the variable separation method has to be solved for the Bessel-function order (the last one is real for propagating waves and purely imaginary for evanescent modes). Moreover, the normalized radial eigenfunction is expressed in terms of the cross-product Bessel function of complex order and its derivatives with respect to the arguments and to this order. Most recently published approaches to the problem have been developed to overcome the difficulty of evaluation of these eigenfunctions and the angular propagation constants with the prescribed degree of accuracy.

According to the related local modes method (Mongiardo, Morini, & Rozzi, 1995), the transverse field is described by means of superposition of the locally straight wave-

Received 7 July 2003; accepted 11 October 2003.
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guide modes that are not orthogonal in the curved region. A method of equivalent transformation of the initial eigenvalue problem into a matrix eigenvalue equation was applied by Weisshaar, Goodnick, and Tripathi (1992). Modal decomposition, which is based on a combination of the same technique and the perturbation analysis (Lewin, Chang, & Kuester, 1977), was used by Gimeno and Guglielmi (1996). Hsu and Anada (1995) realized the calculation of radial eigenfunctions by means of a step-like approximation. An approximation of the bend by elementary mitred bent waveguides was applied by Cornet, Dusseaux, and Chandezon, (1999). Theoretically, the last approach allows simulating nonuniform curved guides with any curvature. Unfortunately, the proposed procedure of arbitrary truncation of the resultant infinite system of linear algebraic equations (SLAE) has not been substantiated.

Commonly, the data obtained were validated by comparison tests or/and computational experiments in the form of "practical convergence of numerical results" only. This is due to great difficulties in rigorously proving the validity of the truncation procedure for the derived infinite SLAE and the stable convergence of numerical approximations to the true solution.

In this paper, we propose the straightforward analytic-numerical approach to the problem of junction discontinuity between a straight and a uniformly curved section of rectangular waveguide. The main objective is to develop an effective matrix model, the correctness of which is rigorously substantiated. Convergence of approximate solutions of the proposed matrix operator equation (MOE) is proved analytically.

The speciality of the model is in the resultant MOE, which is formulated with respect to the unknown reflection operator. To define this matrix operator we use the modal expansions in terms of longitudinal electrical (LE) or magnetic (LM) modes in both the straight and the curved section of the guide. The above-mentioned mathematical difficulties of this eigenmode analysis were overcome in Petrusenko (1983). Analytical study shows that the reflection operator sought for is the Cayley transform of the given accretive operator. The matrix operator technique (MOT) used is a variant of the spectral operator method (Litvinenko & Prosvirnin, 1984; Shestopalov, Kirilenko & Masalov, 1984; Shestopalov & Sirenko, 1989). Besides, the MOT is a straight generalization of the conventional mode-matching technique.

The study of the junction between a straight and a curved guide is not solely of academic interest. It has practical importance as well. The proposed full-wave model should be part of a design tool for microwave CAD/CAM systems. The algorithm developed ensures the determination of the scattering characteristics in a wide range of the geometrical and electrical parameters of the unit.

To keep the mathematical manipulations to a minimum, some useful agreements are used. Assume that $\mathbf{f} \equiv \{f_n(\boldsymbol{r})\}$ and $\mathbf{g} \equiv \{g_n(\boldsymbol{r})\}$ are infinite column vectors of functions ($\boldsymbol{r}$ is a radius vector) and the Hermitian transposition is indicated as $\mathbf{f}^\dagger \equiv (\mathbf{f}^*)^T$, where the superscript T denotes matrix transposition and the asterisk is for complex conjugation. Then the result of inner product $\mathbf{fg}^\dagger = \mathbf{A}$ is the matrix operator-function $\mathbf{A} \equiv \{A_{mn}(\boldsymbol{r}) = f_m g_n^*\}$. Upon integrating $\mathbf{A}$ over the area Ω, we obtain again the matrix operator $(\mathbf{f}, \mathbf{g}^\dagger) \equiv \int_\Omega \mathbf{fg}^\dagger dS = \mathbf{L}$ with the elements $L_{mn} = \int_\Omega A_{mn}(\boldsymbol{r}) dS$.

Geometry of the Problem and Field Representation

The configuration of interest and the frames used are shown in Figure 1. The junction between a straight rectangular waveguide of the width $2a$ and a section of the same guide continuously curved in the H- or E- plane in a circular arc is considered. The regular waveguide regions are marked as I and II (Figure 1). The curved section has the medial

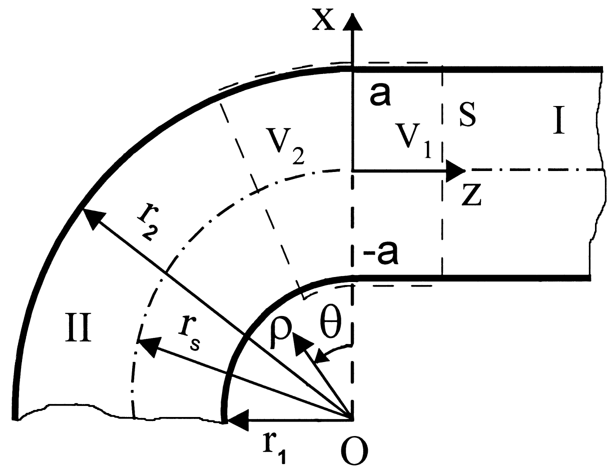

Figure 1. Geometry of the problem and coordinate systems.

radius $r_s = \frac{1}{2}(r_2 + r_1) > a$ and the aperture of junction is $\Omega = \{x \in [-a, a], z = 0\} = \{r \in [r_1, r_2], \theta = 0\}$, where $r = x + r_s$. The metallic walls are assumed to be perfect electric conductors. The waveguides are filled with a homogeneous lossless medium and terminated in matching loads. The convention of time dependence is $\exp(i\omega t)$, and $k = \omega\sqrt{\varepsilon\mu}$ is the wavenumber. A set of LE or LM modes incident upon the junction may be arbitrary, with the understanding that the incident power must be finite.

Let the nonvanishing y-component of the field be $U(\boldsymbol{r})$. This field component $U_j(\boldsymbol{r})$ in the jth partial region will be considered to have the series form

$$U_j(\boldsymbol{r}) = \sum_{p=1}^{\infty} b_p u_p^{(j)}(\boldsymbol{r}) = \mathbf{b}\mathbf{u}_j(\boldsymbol{r}) = \mathbf{u}_j^T(\boldsymbol{r})\mathbf{b}^T, \quad j = 1, 2, \tag{1}$$

in which the row vector of complex coefficients $\mathbf{b} \equiv \{b_p\}$ is *given*, whereas $\mathbf{u}_j(\boldsymbol{r}) \equiv \{u_p^{(j)}(\boldsymbol{r})\}$ is the column vector of *unknown complex-valued functions*. Since vector $\mathbf{b}$ is arbitrary, the standard requirements of the statement of the problem have to be formulated for $u_p^{(j)}(\boldsymbol{r})$, $j = 1, 2$. In particular, each of these functions must satisfy the two-dimensional (2D) Helmholtz equation with the corresponding homogeneous boundary condition on the conducting walls. The solution of the eigenmode problem in the region *II*, which can easily be found via the variable separation method, is given in many papers and books (see, e.g., Cochran & Pecina, 1966; Mahmoud, 1991). We will only consider the modes LM_{m0}, $m = 1, 2, \ldots$, and LE_{n1}, $n = 0, 1, \ldots$, because of their practical importance. In what follows the letters H and E will be connected with the H- and E-plane waveguide bends, which are considered simultaneously.

Separating variables, we obtain the modal expansions for the unknown functions

$$\begin{aligned} {}^s u_p^{(1)}(x, z) &= \varphi_p(x)e^{\gamma_p z} + \sum_{m=(0)1}^{\infty} {}^s X_{pm}\varphi_m(x)e^{-\gamma_m z}; \\ {}^s u_q^{(2)}(\rho, \theta) &= \sum_{n=(0)1}^{\infty} {}^s Y_{qn}\psi_n(\rho)e^{-\nu_n \theta} \end{aligned} \tag{2}$$

by means of the reflection $\mathbf{X} \equiv \{X_{pm}\}$ and transmission $\mathbf{Y} \equiv \{Y_{qn}\}$ matrix operators. By this definition, the last ones are connected with the conventional generalized scattering matrix (Mittra & Lee, 1971). In (2), the left superscript "s" marks a location of the field source in the straight waveguide section. The interchanges $1 \rightleftarrows 2$, $\varphi_m(x) \rightleftarrows \psi_m(\rho)$, and $\gamma_m z \rightleftarrows \nu_m \theta$ give us the modal expansions for the alternative "c"-variant of excitation.

In (2), γ_m and $\nu_n = i\beta_n$ are the propagation constants, the sign of which is chosen in accordance with the condition at infinity for waveguides. Namely, we have

$$\gamma_m = \sqrt{\left(\frac{m\pi}{2a}\right)^2 - \tilde{k}^2}; \quad \tilde{k} = \begin{cases} k & (H) \\ \sqrt{k^2 - \left(\frac{\pi}{l}\right)^2} & (E) \end{cases}, \tag{3}$$

where l is the waveguide height and the angular propagation constant β_n is evaluated either from the modal equation

$$P_{\beta_n}(kr_1, kr_2) = J_{\beta_n}(kr_1)N_{\beta_n}(kr_2) - J_{\beta_n}(kr_2)N_{\beta_n}(kr_1) = 0 \quad (H) \tag{4}$$

or from the dispersion equation

$$\ddot{P}_{\beta_n}(\tilde{k}r_1, \tilde{k}r_2) = J'_{\beta_n}(\tilde{k}r_1)N'_{\beta_n}(\tilde{k}r_2) - J'_{\beta_n}(\tilde{k}r_2)N'_{\beta_n}(\tilde{k}r_1) = 0. \quad (E) \tag{5}$$

Here J_ν and N_ν are the Bessel and, correspondingly, Neumann function; the prime denotes the derivative with respect to the argument. If the wavenumber $\tilde{k}$ is real, then each of these equations has the finite number of the real β-zeros and an infinite set of purely imaginary ones.

The normalized transverse eigenfunctions are

$$\varphi_m(x) = \begin{cases} \frac{1}{\sqrt{a}} \sin\left(\frac{m\pi(x+a)}{2a}\right), & m = 1, 2, \ldots, (H) \\ \sqrt{\frac{2-\delta_{m0}}{2a}} \cos\left(\frac{m\pi(x+a)}{2a}\right), & m = 0, 1, \ldots, (E) \end{cases} \tag{6}$$

$$\psi_n(\rho) = \begin{cases} P_{\beta_n}(k\rho, kr_2)\left[\frac{\rho}{2\beta_n}\frac{\partial P_{\beta_n}(k\rho, kr_2)}{\partial\rho}\frac{\partial P_{\beta_n}(k\rho, kr_2)}{\partial\beta}\right]^{-1/2}_{\rho=r_1}, & n = 1, 2, \ldots, (H) \\ \dot{P}_{\beta_n}(\tilde{k}\rho, \tilde{k}r_2)\left[-\frac{\rho}{2\beta_n}\frac{\partial \dot{P}_{\beta_n}(\tilde{k}\rho, \tilde{k}r_2)}{\partial\rho}\frac{\partial \ddot{P}_{\beta_n}(\tilde{k}\rho, \tilde{k}r_2)}{\partial\beta}\right]^{-1/2}_{\rho=r_1}, & n = 0, 1, \ldots, (E) \end{cases} \tag{7}$$

where δ_{mn} is the Kronecker delta and the derivative of the cross-product Bessel function is denoted as

$$\dot{P}_\beta(\tilde{k}\rho, \tilde{k}r_2) = J_\beta(\tilde{k}\rho)N'_\beta(\tilde{k}r_2) - J'_\beta(\tilde{k}r_2)N_\beta(\tilde{k}\rho). \tag{8}$$

These eigenfunctions, being solutions of the Sturm-Liouville problem, constitute the complete orthonormal systems. Thus, for the real-valued functions (6), (7), which are collected into the column vectors $\boldsymbol{\varphi}(x) = \{\varphi_m(x)\}$ and $\boldsymbol{\psi}(r) = \{\psi_n(r)\}$, we have

$$(\boldsymbol{\varphi}, \boldsymbol{\varphi}^T) = \mathbf{I}, \quad (\boldsymbol{\psi}, \boldsymbol{\psi}^T r^{-1}) = \mathbf{I}, \tag{9}$$

where $\mathbf{I}$ is the identity.

From (1) and (2), the usual modal expansions of unknown field components formally follow

$$U_1(x,z) = U_{inc}(x,z) + \sum_{m=1}^{\infty} c_m \varphi_m(x) e^{-\gamma_m z},$$
$$U_{inc}(x,z) = \sum_{p=1}^{\infty} b_p \varphi_p(x) e^{\gamma_p z}, \tag{10}$$
$$U_2(\rho,\theta) = \sum_{n=1}^{\infty} t_n \psi_n(\rho) e^{-\nu_n \theta}.$$

Here U_{inc} signifies the set of incident modes. To ensure finiteness of the energy stored in any volume inside the guide we require that $\mathbf{b}, \mathbf{c}, \mathbf{t} \in h_\gamma$, where the space of sequences $h_\gamma \subset \ell_2$ is defined as

$$h_\gamma \overset{def}{=} \left\{ \mathbf{b} : \sum_{m=1}^{\infty} |\gamma_m||b_m|^2 = \|\mathbf{b}\|_\gamma^2 < \infty \right\}. \tag{11}$$

(Note that h_γ is equivalent to the Hilbert space $\tilde{\ell}_2$, which is usually used for the same purpose; Shestopalov & Sirenko, 1989).

In (10) the row vectors of unknown reflection and transmission coefficients are

$$\mathbf{c} = \mathbf{bX}, \;\; \mathbf{t} = \mathbf{bY}. \tag{12}$$

The properties of matrix operators $\mathbf{X}, \mathbf{Y}$: $h_\gamma \to h_\gamma$ are of decisive importance for substantiation of the proposed model. From the physical standpoint, these operators have to be continuous ones. Moreover, taking into account the physical meaning of (12) it must be $\|\mathbf{bX}\|_\gamma \le \|\mathbf{b}\|_\gamma$ and $\|\mathbf{bY}\|_\gamma \le \|\mathbf{b}\|_\gamma$; i.e., the reflection and transmission operators are contractions. These properties will be established rigorously in the next sections.

Reciprocity and Conservation-of-Energy Principles in Operator Matrix Form

In accordance with the MOT, the basic electromagnetic principles have to be formulated in the operator form. For this purpose, we define the standardized operators

$$\begin{gathered} {}^s\mathbf{R} = \mathbf{I}_\gamma^{-1/2}\,{}^s\mathbf{X}\mathbf{I}_\gamma^{1/2},\; {}^s\mathbf{T} = \mathbf{I}_\gamma^{-1/2}\,{}^s\mathbf{Y}\mathbf{I}_\nu^{1/2}, \\ {}^c\mathbf{R} = \mathbf{I}_\nu^{-1/2}\,{}^c\mathbf{X}\mathbf{I}_\nu^{1/2},\; {}^c\mathbf{T} = \mathbf{I}_\nu^{-1/2}\,{}^c\mathbf{Y}\mathbf{I}_\gamma^{1/2} \end{gathered} \tag{13}$$

in the Hilbert space ℓ_2. Here we introduce the matrix operator $\mathbf{I}_\beta^\sigma \equiv \{\beta_m^\sigma \delta_{mn}\}$. For simplicity, the critical values of the wavenumber are excluded (i.e., $\gamma_m \neq 0$, $\nu_n \neq 0$ $\forall m, n$).

The set of the equalities needed follows from the fact that the row vector $\mathbf{b}$ in (1) is common for two waveguide regions. Thus, the condition of continuity of the tangential electric and magnetic fields at the aperture Ω gives us the set of implications

$$\begin{cases} U_1 = U_2, \\ \dfrac{\partial U_1}{\partial \boldsymbol{n}} = \dfrac{\partial U_2}{\partial \boldsymbol{n}} \end{cases}, \qquad \boldsymbol{r} \in \Omega \Rightarrow \begin{cases} \mathbf{b}(\mathbf{u}_1 - \mathbf{u}_2)|_\Omega = 0, \\ \mathbf{b}\left(\dfrac{\partial \mathbf{u}_1}{\partial \boldsymbol{n}} - \dfrac{\partial \mathbf{u}_2}{\partial \boldsymbol{n}}\right)\Big|_\Omega = 0, \end{cases} \Rightarrow \forall \mathbf{b}$$

$$\Rightarrow \begin{cases} u_p^{(1)}(x,0) = u_p^{(2)}(r,0), \\ \dfrac{\partial u_q^{(1)}}{\partial z}\Big|_{z=0} = -\dfrac{1}{r}\dfrac{\partial u_q^{(2)}}{\partial \theta}\Big|_{\theta=0}, \quad x, r \in \Omega, \ \forall p, q, \end{cases} \tag{14}$$

where $\boldsymbol{n}$ is a normal to the junction plane. (It is pertinent to note that the last system in (14) corresponds to the mode-matching ideology.) Again, the continuity of the oscillating part of the power flow at the aperture and the Lorentz reciprocity theorem leads to the matrix relations

$$\begin{aligned} \left({}^{s(c)}\mathbf{u}_1, \frac{\partial^{s(c)}\mathbf{u}_1^T}{\partial \boldsymbol{n}}\right) &= \left({}^{s(c)}\mathbf{u}_2, \frac{\partial^{s(c)}\mathbf{u}_2^T}{\partial \boldsymbol{n}}\right), \\ \left({}^{s(c)}\boldsymbol{u}_1, \frac{\partial^{c(s)}\mathbf{u}_1^T}{\partial \boldsymbol{n}}\right) &= \left({}^{s(c)}\mathbf{u}_2, \frac{\partial^{c(s)}\mathbf{u}_2^T}{\partial \boldsymbol{n}}\right), \end{aligned} \tag{15}$$

respectively. The second pair of matrix equalities is

$$\begin{aligned} \left({}^{s(c)}\mathbf{u}_1, \frac{\partial^{s(c)}\mathbf{u}_1^\dagger}{\partial \boldsymbol{n}}\right) &= \left({}^{s(c)}\mathbf{u}_2, \frac{\partial^{s(c)}\mathbf{u}_2^\dagger}{\partial \boldsymbol{n}}\right), \\ \left({}^{s(c)}\mathbf{u}_1, \frac{\partial^{c(s)}\mathbf{u}_1^\dagger}{\partial \boldsymbol{n}}\right) &= \left({}^{s(c)}\mathbf{u}_2, \frac{\partial^{c(s)}\mathbf{u}_2^\dagger}{\partial \boldsymbol{n}}\right), \end{aligned} \tag{16}$$

where the first relation is a consequence of continuity of the complex power flow. The last equality in (16) is valid, in particular, for the abrupt junctions such as a waveguide bifurcation, step discontinuities, a break of guide curvature, etc. The proof of a corresponding lemma is given in Appendix A.

Substituting (2) into (15) and making use of the orthogonality conditions (9), we get the reciprocity relations

$${}^{s(c)}\mathbf{R}^T = {}^{s(c)}\mathbf{R},\ {}^{s}\mathbf{T}^T = {}^{c}\mathbf{T},\ {}^{s}\mathbf{R}^{s}\mathbf{T} + ({}^{c}\mathbf{R}^{c}\mathbf{T})^T = \mathbf{0},\ {}^{s(c)}\mathbf{R}^2 + {}^{s(c)}\mathbf{T}^{c(s)}\mathbf{T} = \mathbf{I}. \tag{17}$$

Along similar lines, from (16) we find the power conservation law (PCL). For its convenient representation, let us introduce the diagonal unitary operators

$$\mathbf{U}_P = \mathbf{I}_\gamma^{-1/4}\mathbf{I}_{\gamma^*}^{1/4},\ \mathbf{U}_Q = \mathbf{I}_\nu^{-1/4}\mathbf{I}_{\nu^*}^{1/4}, \tag{18}$$

where P and Q symbolize the numbers of propagating waves in the regions I and II, respectively. (It is well known that $P \leq Q$ for H-bend and $P \geq Q$ for E-bend; see, e.g., Cochran & Pecina, 1966). Then in terms of the unitarily equivalent operators

$$
\begin{aligned}
{}^{s}\mathbf{R}_U &= \mathbf{U}_P^{*}\,{}^{s}\mathbf{R}\mathbf{U}_P,\ {}^{s}\mathbf{T}_U = \mathbf{U}_P^{*}\,{}^{s}\mathbf{T}\mathbf{U}_Q,\\
{}^{c}\mathbf{R}_U &= \mathbf{U}_Q^{*}\,{}^{c}\mathbf{R}\mathbf{U}_Q,\ {}^{c}\mathbf{T}_U = \mathbf{U}_Q^{*}\,{}^{c}\mathbf{T}\mathbf{U}_P,
\end{aligned}
\tag{19}
$$

the sought-for PCL is

$$
\begin{aligned}
(\mathbf{I} + {}^{s(c)}\mathbf{R}_U)(\mathbf{I} - {}^{s(c)}\mathbf{R}_U^{*}) &= {}^{s(c)}\mathbf{T}_U\,{}^{c(s)}\mathbf{T}_U^{*},\\
(\mathbf{I} + {}^{s(c)}\mathbf{R}_U)\,{}^{s(c)}\mathbf{T}_U^{*} &= {}^{s(c)}\mathbf{T}_U(\mathbf{I} - {}^{c(s)}\mathbf{R}_U^{*}).
\end{aligned}
\tag{20}
$$

In addition, the last relation in (17) can also be represented as

$$
(\mathbf{I} + {}^{s(c)}\mathbf{R}_U)(\mathbf{I} - {}^{s(c)}\mathbf{R}_U) = {}^{s(c)}\mathbf{T}_U\,{}^{c(s)}\mathbf{T}_U. \tag{21}
$$

To generalize the found fundamental relations (17), (20), and (21), let us consider the operator matrices

$$
\mathbf{S} = \begin{pmatrix} {}^{s}\mathbf{R} & {}^{s}\mathbf{T} \\ {}^{c}\mathbf{T} & {}^{c}\mathbf{R} \end{pmatrix}, \quad \mathbf{U} = \begin{pmatrix} \mathbf{U}_P & \mathbf{0} \\ \mathbf{0} & \mathbf{U}_Q \end{pmatrix}, \quad \mathbf{S}_U = \mathbf{U}^{*}\mathbf{S}\mathbf{U}, \tag{22}
$$

where $\mathbf{S}_U$ is a unitarily equivalent matrix to the generalized scattering matrix $\mathbf{S}$. Then the reciprocity conditions (17), (21) take the simple form

$$
\mathbf{S} = \mathbf{S}^{T} = \mathbf{S}^{-1}, \qquad \mathbf{S}_U = \mathbf{S}_U^{-1}, \tag{23}
$$

and the relations (20) give us the generalized power conservation law (GPCL)

$$
(\mathbf{I} + \mathbf{S}_U)(\mathbf{I} - \mathbf{S}_U^{*}) = \mathbf{0}. \tag{24}
$$

The significance of the equalities (23), (24) is that they uniquely determine the spectral properties of the reflection and transmission operators.

Matrix Model and Its Characteristic Properties

Consider first the sought-for solution for the “s”-variant of excitation. For this case, the superscript will be omitted for simplicity.

Substituting (2) into (14), we obtain the system of matrix equations

$$
\begin{cases}
(\mathbf{I} + \mathbf{X})\boldsymbol{\varphi}(x) = \mathbf{Y}\boldsymbol{\psi}(r),\\
(\mathbf{I} - \mathbf{X})\mathbf{I}_{\gamma}\boldsymbol{\varphi}(x) = \mathbf{Y}\mathbf{I}_{\nu}\boldsymbol{\psi}(r)r^{-1}.
\end{cases}
\tag{25}
$$

According to the Galerkin method, the operators $(\bullet, \boldsymbol{\varphi}^{T})$ and $(\bullet, \boldsymbol{\psi}^{T})$ are applied to the first and, respectively, to the second subsystem. With a simple algebra, the resultant system can be shown to take the form

$$
\begin{cases}
\mathbf{I} + \mathbf{R} = (\mathbf{I} - \mathbf{R})\mathbf{D}\mathbf{D}^{T},\\
\mathbf{T} = (\mathbf{I} - \mathbf{R})\mathbf{D},
\end{cases}
\tag{26}
$$

where

$$\mathbf{D} = \mathbf{I}_{\gamma}^{1/2}(\boldsymbol{\varphi}, \boldsymbol{\psi}^T)\mathbf{I}_{\nu}^{-1/2} \tag{27}$$

is the infinite matrix of "function moments." From the system (26), it formally follows that

$$\mathbf{R} = (\mathbf{DD}^T - \mathbf{I})(\mathbf{DD}^T + \mathbf{I})^{-1}, \tag{28}$$

$$\mathbf{DD}^T = (\mathbf{I} - \mathbf{R})^{-1}(\mathbf{I} + \mathbf{R}). \tag{29}$$

These relations are the well-known Cayley transforms of operators (see, e.g., Richtmyer, 1978).

Another form of the sought-for solution (28) is

$$\mathbf{R} = \frac{1}{2}\mathbf{B}\left(\mathbf{I} + \frac{1}{2}\mathbf{B}\right)^{-1}, \tag{30}$$

where $\mathbf{B} = \mathbf{DD}^T - \mathbf{I}$. In Appendix B, it is proved that $\mathbf{DD}^T$ is bounded, whereas the operator $\mathbf{B}$ is a compact one. Then from (30) it follows that $\mathbf{R}$ is a compact operator too.

To obtain analogous results for the alternative location of the field source in the region II, the substitutions

$$\mathbf{D} \to \mathbf{D}^{-1} = \mathbf{I}_{\nu}^{1/2}(r^{-1}\boldsymbol{\psi}, \boldsymbol{\varphi}^T)\mathbf{I}_{\gamma}^{-1/2} \tag{31}$$

must be made into the formulae (28), (29).

The fundamental properties of the operators under consideration can be recognized from the localization of their spectrum. According to (29), the linear-fractional mapping $W(\lambda) = (1 + \lambda)(1 - \lambda)^{-1}$ of spectrum $\sigma(\mathbf{R})$ to the spectrum $\sigma(\mathbf{DD}^T)$ takes place. From the PCL (20), it follows that the $\sigma(\mathbf{R})$ lies in the interior of the unit disk and, hence, the numbers ± 1 do not belong to this spectrum (the details are given in Appendix C). Then the conformal transformation $\sigma(\mathbf{DD}^T) = W(\sigma(\mathbf{R}))$ maps this interior of unit disk on the right half-plane and $0 \notin \sigma(\mathbf{DD}^T)$. Therefore, the spectrum of the operator $\mathrm{Re}\mathbf{DD}^T = \frac{1}{2}[\mathbf{DD}^T + (\mathbf{DD}^T)^*]$ lies on the positive real axis and, hence, $\mathbf{DD}^T$ is an accretive operator. (By the definition, a bounded operator $\boldsymbol{L}$ is an accretive one if its real part is a positive operator; *Encyclopaedia of Mathematics*, 1995. Note that it is common for mathematical literature to consider the dissipative operator $i\boldsymbol{L}$ instead of the accretive one.) Thus, the formulae (28) and (29) form the well-studied pair of Cayley transforms for the contraction $\mathbf{R}$ and the accretive $\mathbf{DD}^T$ operator (*Encyclopaedia of Mathematics*, 1995).

The correctness of the solutions (28), (30) depends on the properties of operator

$$\mathbf{A} \equiv (\mathbf{I} + \mathbf{DD}^T)^{-1} = \frac{1}{2}\left(\mathbf{I} + \frac{1}{2}\mathbf{B}\right)^{-1} = \frac{1}{2}(\mathbf{I} - \mathbf{R}). \tag{32}$$

As a corollary of the Cayley transform theory and GPCL (24), $\mathbf{A}(k)$ is a regular operator-function for every value of the wavenumber k. Moreover, from the estimate $\|\mathbf{A}\| \leq \frac{1}{2}(1 + \|\mathbf{R}\|) \leq 1$ it follows that this matrix operator is a contraction.

Thus, the matrix model proposed is complete and consistent.

Convergence of Approximate Solutions

Let the series in (2) be truncated and the retained number of modes in the straight and curved guide be *M* and *N*, respectively. Starting from the imposed condition of equality of the finite sums instead of (14) and (25), similar reasoning can be repeated. As a result, the main conclusions above remain valid.

Thus, let $\widehat{\mathbf{DD}^T}$ be the matrix of the truncated equations (26); then the approximate solution takes the form

$$\hat{\mathbf{R}} = \hat{\mathbf{A}}(\widehat{\mathbf{DD}^T} - \mathbf{P}^M), \tag{33}$$

where

$$\hat{\mathbf{A}} = (\widehat{\mathbf{DD}^T} + \mathbf{P}^M)^{-1} = \frac{1}{2}(\mathbf{P}^M - \hat{\mathbf{R}}) \tag{34}$$

and

$$\mathbf{P}^M \equiv \left\{ P^M_{mn} = \sum_{p=(0)1}^{M} \delta_{mp}\delta_{pn},\ \forall m, n \right\};$$

the latter is also an ortho-projector. The matrix (34) exists and is well conditioned because, again, $\pm 1 \notin \sigma_p(\hat{\mathbf{R}})$, $-1 \notin \sigma_p(\widehat{\mathbf{DD}^T})$, and $\hat{\mathbf{A}}$ is a contraction for all values of M, N. Its condition number is bounded above as

$$\text{cond}\ (\hat{\mathbf{A}}) = \|\hat{\mathbf{A}}\| \|\mathbf{P}^M + \widehat{\mathbf{DD}^T}\| \leq 1 + \|\widehat{\mathbf{DD}^T}\|. \tag{35}$$

With the help of (32), (34), the error of approximate solution

$$\begin{aligned} \hat{\mathbf{R}} - \mathbf{P}^M\mathbf{R} &= 2(\mathbf{P}^M\mathbf{A} - \hat{\mathbf{A}}) = -2\mathbf{P}^M\mathbf{A}\Xi\hat{\mathbf{A}}, \\ \Xi &= \mathbf{P}^M\mathbf{DD}^T - \widehat{\mathbf{DD}^T} \end{aligned} \tag{36}$$

is expressed by means of the error of approximation of the given accretive operator

$$\Xi = (\mathbf{P}^M\mathbf{B} + \mathbf{P}^M) - \mathbf{P}^M(\widehat{\mathbf{DD}^T} - \mathbf{P}^N) - \mathbf{P}^M\mathbf{P}^N = \mathbf{P}^M(\mathbf{I} - \mathbf{P}^N) + \mathbf{P}^M\tilde{\mathbf{B}}. \tag{37}$$

Here $\tilde{\mathbf{B}}$ is a part of matrix operator $\mathbf{B}$ such that $\|\tilde{\mathbf{B}}\| \to 0$ when $N \to \infty$ (see Appendix B).

It results from (37) that convergence of the approximations can be uniform or nonuniform depending on the ratio M/N. Since for a sufficiently large (but finite) N we obtain the estimate

$$\Xi = \begin{cases} \mathbf{P}^M(\mathbf{I} - \mathbf{P}^N) + \boldsymbol{\varepsilon}, & \|\boldsymbol{\varepsilon}\| \ll 1,\ M > N, \\ \mathbf{P}^M\tilde{\mathbf{B}} \equiv \boldsymbol{\delta}, & \|\boldsymbol{\delta}\| \ll 1,\ M \leq N \end{cases} \Rightarrow \left. \begin{matrix} O(1) \\ \|\boldsymbol{\delta}\| \end{matrix} \right\} = \|\Xi\|, \tag{38}$$

then only when $M \leq N$ does the sought-after uniform convergence take place:

$$\|\hat{\mathbf{R}} - \mathbf{P}^M\mathbf{R}\| \leq 2\|\Xi\| \to 0, \quad M \to \infty. \tag{39}$$

In a similar manner, we find the condition $M \geq N$ of the uniform convergence in the case of the second "c"-variant of excitation.

For pointwise (or strong) convergence of solution (33), this phenomenon of relative convergence is absent. Indeed, from (36), (37) it follows that

$$\|\hat{\mathbf{R}}\mathbf{b}^T - \mathbf{P}^M\mathbf{R}\mathbf{b}^T\| \leq 2\|\mathbf{P}^M(\mathbf{I}-\mathbf{P}^N)\mathbf{c}\| + 2\|\mathbf{P}^M\tilde{\mathbf{B}}\|\|\mathbf{b}^T\|, \tag{40}$$

where the row vector $\mathbf{b} \in \ell_2$ is arbitrary and $\mathbf{c} = \hat{\mathbf{A}}\mathbf{b}^T$. Because $\|\mathbf{P}^M(\mathbf{I}-\mathbf{P}^N)\mathbf{c}\| \to 0$ $\forall\mathbf{c} \in \ell_2$, when $M > N$, $N \to \infty$, we arrive at

$$\|\hat{\mathbf{R}}\mathbf{b}^T - \mathbf{P}^M\mathbf{R}\mathbf{b}^T\| \to 0, \quad M, N \to \infty\,\forall M/N. \tag{41}$$

The traditionally checked coordinate-wise (or weak) convergence follows immediately from (41). Again, this is an unconditional convergence for the problem.

Finally, let $\mathrm{cond}(\mathbf{P}^M\mathbf{A}) \equiv \|\mathbf{P}^M\mathbf{A}\|\|\mathbf{P}^M\mathbf{A}^{-1}\|$; then, using (32) and (34)–(36), we find

$$|\mathrm{cond}(\mathbf{P}^M\mathbf{A}) - \mathrm{cond}(\hat{\mathbf{A}})| < \mathrm{const} \cdot \|\boldsymbol{\Xi}\|,$$

and, hence, the sequence of condition numbers (35) is uniformly bounded one when $M, N \to \infty$.

Numerical Results

The developed model (33) is suitable for effective realization in the modern systems of matrix computations. For this purpose, the highly accurate calculations of the cross-product Bessel function with complex index and its derivatives with respect to the argument and index have been carried out. These algorithms are based on the rapidly convergent series representations of the Neumann type (Petrusenko, 1983; Petrusenko & Dmitryuk, 1986). The angular propagation constants for both the dominant and evanescent modes were calculated by the Newton method with the help of the higher order approximations (Petrusenko, 2001). The integral in (27) was conveniently computed based on numerical techniques.

The problem of uniform convergence of matric solutions and associated phenomenon of relative convergence are the subject for a separate work. Here, we demonstrate only the numerical efficiency of the proposed approach.

Figures 2a–2d illustrate convergence of the numeric algorithm and the condition number (35) subjects to the truncation numbers M, N for the "s"-case of excitation by the dominant mode of a unit amplitude. Thus, $\mathbf{b} = \{1; 0; \ldots ; 0\}$. For the second variant of excitation, the dependences are closely related to the presented lines (after the interchanging $M \rightleftarrows N$). In each figure, we cite the greater values found for two sets of modes.

The typical behavior of the relative error of approximations is shown in Figures 2a and 2b. In computing these error functions, the data corresponding to $M = 30$, $N = 60$ have been fixed as the reference values. One can see that the error decreases rapidly when $N \geq 2M$. The rate of decay of the error function can be taken as a cost of the algorithm. Thus, it is enough to take few equations to achieve the reasonable accuracy. Below, the evaluation of physical characteristics is performed with $N = 25$ and the 10×10 SLAE. For this size of the truncated matrix, the relative error is less than $5 \cdot 10^{-5}$.

Figures 2c and 2d show typical values of the condition number, which are in close proximity to unity when $N \geq M$.

In Table 1, the calculated values of the reflected and transmitted mode power are compared with the data obtained by Bates (1969). In that method-of-moments solution,

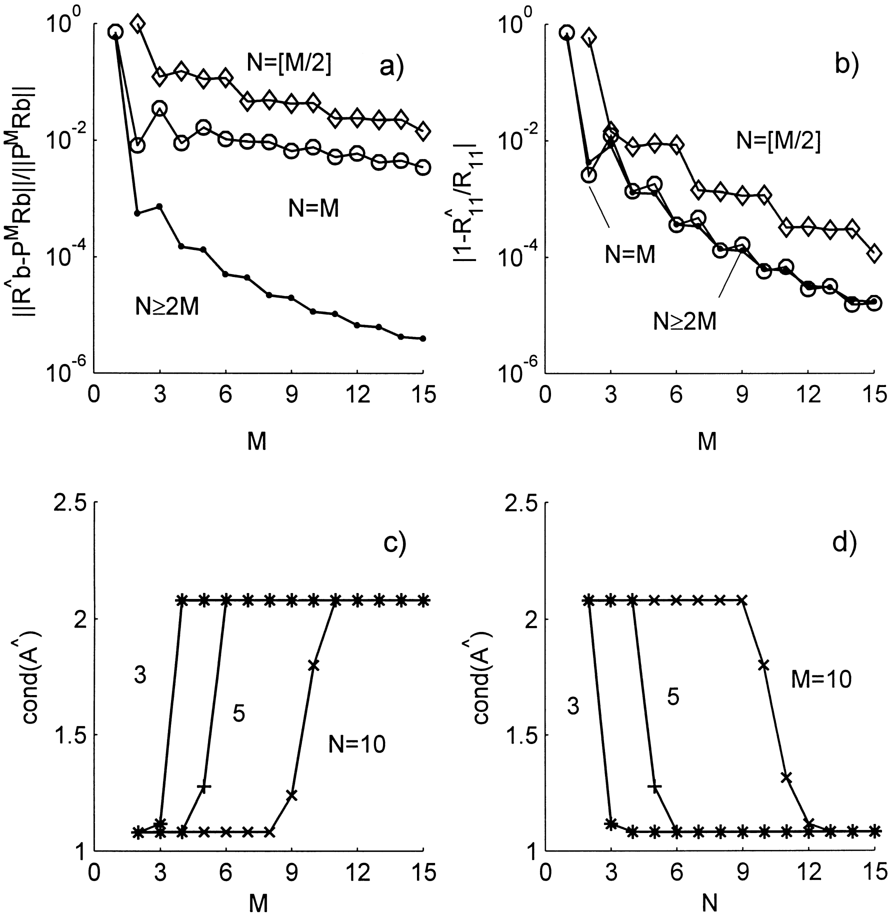

Figure 2. Convergence of the numerical approximations and the condition number as a function of the number of modes M, N remaining after truncation for $a/\lambda = 0.36$, $r_1/r_2 = 0.51644$. The relative error of (a) first column of the reflection matrix and (b) reflection coefficient of the dominant mode against the matrix truncation number M. Diamonds: $N = entire(M/2)$; circles: $N = M$; points: $N \geq 2M$. The condition numbers of the matrix $\hat{\mathbf{A}}$ of the equation (34) against (c) the matrix truncation number M and (d) the number N of modes in the curved guide. Asterisks: $N(M) = 3$; plus signs: $N(M) = 5$; crosses: $N(M) = 10$.

the Liouville-Green approximations were employed for evaluation of the functions (4), (5), and (8) with large-in-modulus, purely imaginary indices (see also Wu, 1987). In spite of the use of these nonuniform asymptotic expansions, the agreement is fairly good.

The power coupling in the first four modes that propagate in the E-plane straight and curved waveguides is shown in Figure 3. The obtained numeric results are compared with the data presented by Bates (1969) and Wu (1987). They agree well with our data within drawing precision. The discrepancies are most conspicuous near the cut-off points, where the above-mentioned asymptotic approximation of the cross-product Bessel function becomes a rough estimate. Note also that a peak of the reflected power for the mode LE_{01} in the straight guide was lost near the point $l/\lambda = 3.3$, where l is the waveguide height.

Table 1
Intermodal coupling for a square guide with $r_1/r_2 = 0.516441$

Incident mode	Reflected mode power: Bates (1969)	Reflected mode power: This work	Transmitted mode power: Bates (1969)	Transmitted mode power: This work
		$\tilde{k}\alpha = 1.19\pi$		
LM^s_{10}	$LM^s_{10} \approx 10^{-7}$	$9.677 \cdot 10^{-8}$	$LM^c_{10} = 9.52174 \cdot 10^{-1}$	$9.531 \cdot 10^{-1}$
	$LM^s_{20} = 1.2 \cdot 10^{-5}$	$1.247 \cdot 10^{-5}$	$LM^c_{20} = 4.7815 \cdot 10^{-2}$	$4.693 \cdot 10^{-2}$
LE^s_{01}	$LE^s_{01} = 10^{-6}$	$1.312 \cdot 10^{-6}$	$LE^c_{01} = 6.18860 \cdot 10^{-1}$	$6.203 \cdot 10^{-1}$
	$LE^s_{11} = 4.6 \cdot 10^{-5}$	$4.701 \cdot 10^{-5}$	$LE^c_{11} = 3.76074 \cdot 10^{-1}$	$3.749 \cdot 10^{-1}$
	$LE^s_{21} = 2.2 \cdot 10^{-5}$	$2.421 \cdot 10^{-5}$	$LE^c_{21} = 4.997 \cdot 10^{-3}$	$4.777 \cdot 10^{-3}$
		$\tilde{k}\alpha = 1.79\pi$		
LM^s_{10}	$LM^s_{10} = 10^{-9}$	$7.731 \cdot 10^{-10}$	$LM^c_{10} = 7.36720 \cdot 10^{-1}$	$7.370 \cdot 10^{-1}$
	$LM^s_{20} = 10^{-6}$	$1.003 \cdot 10^{-7}$	$LM^c_{20} = 2.58699 \cdot 10^{-1}$	$2.584 \cdot 10^{-1}$
	$LM^s_{30} \approx 10^{-7}$	$2.215 \cdot 10^{-8}$	$LM^c_{30} = 4.581 \cdot 10^{-3}$	$4.553 \cdot 10^{-3}$
LE^s_{01}	$LE^s_{01} \approx 10^{-7}$	$2.562 \cdot 10^{-7}$	$LE^c_{01} = 4.56751 \cdot 10^{-1}$	$4.568 \cdot 10^{-1}$
	$LE^s_{11} = 5 \cdot 10^{-6}$	$5.123 \cdot 10^{-6}$	$LE^c_{11} = 3.84346 \cdot 10^{-1}$	$3.845 \cdot 10^{-1}$
	$LE^s_{21} = 10^{-6}$	$9.711 \cdot 10^{-7}$	$LE^c_{21} = 1.53564 \cdot 10^{-1}$	$1.534 \cdot 10^{-1}$
	$LE^s_{31} = 2.7 \cdot 10^{-5}$	$2.737 \cdot 10^{-5}$	$LE^c_{31} = 5.306 \cdot 10^{-3}$	$5.281 \cdot 10^{-3}$

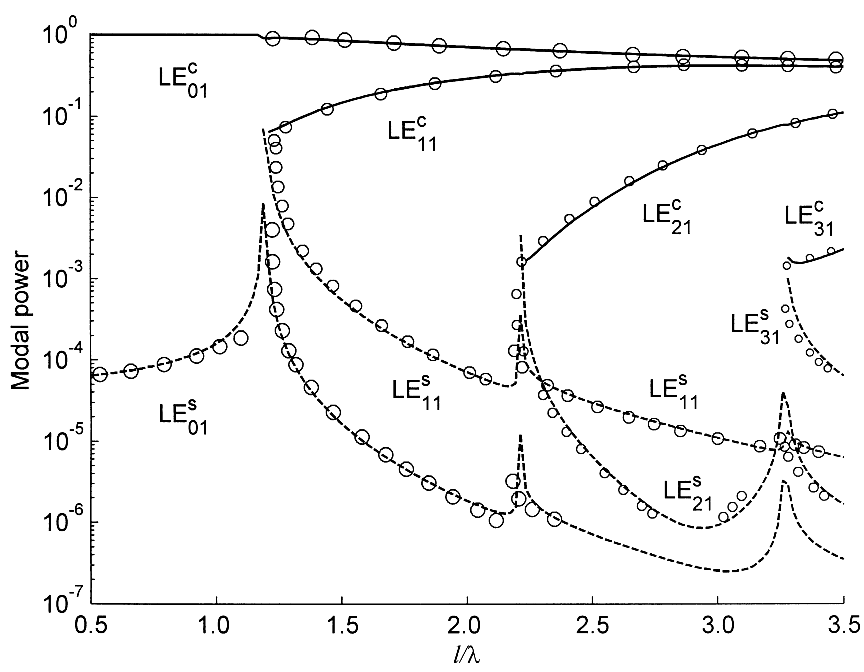

Figure 3. Intermodal coupling at E-plane bend with $r_1/r_2 = 0.53445$ and the normalized waveguide height $l/a = 4.3$ for the incident mode LE^s_{01} of a unit amplitude. Solid line is for a curved guide; dashed line is for a straight guide. Circles indicate the data of Wu (1987) for the LE_{01} (large circles), LE_{11} (middle circles), LE_{21} (small circles), and LE_{31} (the smallest circles) modes.

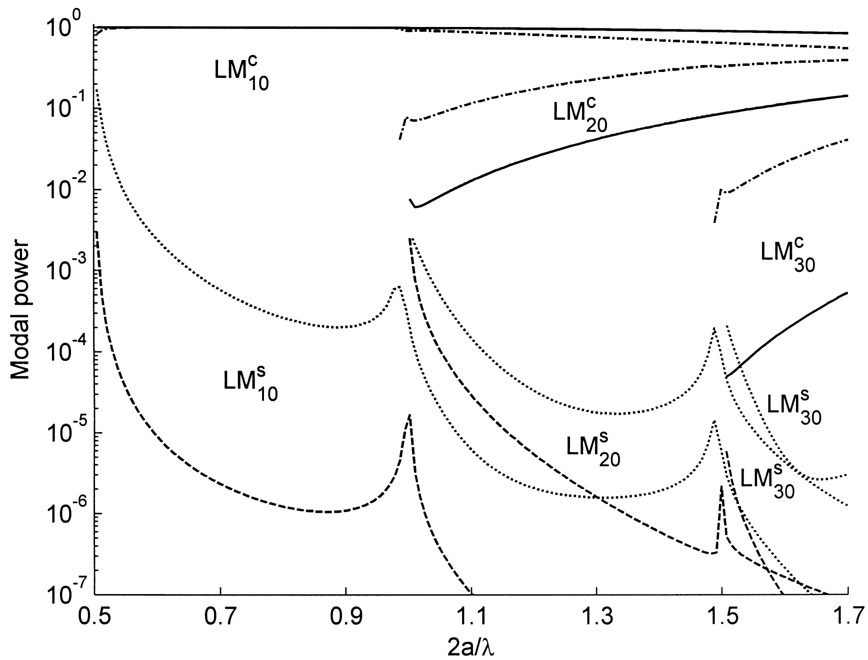

Figure 4. Intermodal coupling at H-plane's very sharp bend with $r_1/r_2 = 0.1$ (dotted and dash-dotted lines) and at the gradual one with $r_1/r_2 = 0.6416$ (solid and dashed lines). The incident mode, of a unit amplitude, is LM^s_{10}. Solid and dash-dotted lines are for a curved guide; dashed and dotted lines are for a straight guide.

Figure 4 shows intermodal coupling at the H-plane's very sharp bend and more gradual one. The latter corresponds to the largest value of curvature for the usually manufactured bends of standard rectangular waveguides WR42-WR03. As seen, for this bend the transmitted power of higher modes is some orders of magnitude less than that for the more sharp bend.

Discussion and Conclusions

Engineering practice is to reduce the initial boundary value problem to an infinite SLAE of the second kind with respect to a single sequence of unknowns. For example, such a matrix model is a typical one for the conventional method of moments. As a rule, this SLAE is not regular or quasi-regular, and its matrix operator does not possess such a strong feature as compactness. In such a situation, it is hard to justify rigorously the correctness of the matrix equation, the validity of the truncation procedure, and the stable convergence of numeric approximations to the true solution. However, in modern research, the substantiation of the model has become important because of the phenomenon of relative convergence, which is as a rule, inherent the method-of-moments solution. As indicated above, the difficulties may be obviated using a MOE instead of the SLAE. For the problem under consideration, such a MOE is the Cayley transformation (28).

It follows from the presented analysis that there exists a certain "duality" between the sought-for reflection operator $\mathbf{R}$ and the given infinite matrix $\mathbf{DD}^T$. Namely, a priori we are in the dark about the values of elements of $\mathbf{R}$ and, at the same time, the important properties of this operator (such as the spectrum localization, etc.) can be easily established. To the contrary, the entries of the matrix $\mathbf{DD}^T$ are known, although the knowledge of its operator properties is insufficient. The interrelation between these two operators in the form of MOE allows us to obtain the sought-after information.

Note that the properties of the reflection operator follow directly from the fundamental physical principles, and hence $\mathbf{R}$ must of necessity be a contraction for every wave scattering problem.

It is commonplace that the conservation of complex power is not a proper measure of the accuracy of approximations. It has long been found for the mode-matching technique that the PCL is automatically satisfied for all the numerical solutions and is independent of the remaining number of modes. Therefore, in many papers and books, one can read that power conservation is only a check of algebra, computer programming, and roundoff error. In the presented formulation, however, the obedience of each approximation to the GPCL (24) "automatically" guarantees the nonsingularity of the matrix of the truncated MOE and the stable computations for any number of matrix truncations.

Thus, the problem of an accurate and rigorous analysis of the junction between a straight and a continuously curved section of rectangular waveguide has been solved using the matrix operator technique. The fundamental properties of the generalized scattering matrix of the discontinuity have been formulated in the operator form; to do this, the second Lorentz lemma for complex power has been used. It has been shown that the sought-for reflection operator is the Cayley transform of the given accretive operator. This proves that the matrix model developed is a well-posed one, and hence it allows robust computations. The convergence of the approximate solutions was proved analytically. It has been shown that the relative convergence takes place if the uniform convergence is considered. However, for other types of convergence this phenomenon is absent. The results of the computations lend support to the validity of the theoretical conclusions reached. It was found that the computational efficiency depended on the relative numbers of the retained modes. Namely, a numeric solution converges more rapidly to the correct answer when the proper ratio is used. The results reveal that it must be $N \geq 2M$ for efficient computations in the case of excitation of the straight guide section and $M \geq 2N$ otherwise. The agreement between the presented results and the known data is documented.

The approach proposed can be efficiently applied to the rigorous analysis of any abrupt waveguide junctions.

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Appendix A

According to two variants of excitation, in the volume $V = V_1 \cup V_2$ bounded by a closed surface S (Figure 1), there exist two fields ${}^s\mathbf{E}$, ${}^s\mathbf{H}$ and ${}^c\mathbf{E}$, ${}^c\mathbf{H}$. Reasoning as in the proof of the Lorentz reciprocity theorem (see, e.g., Collin, 1991) but for the sets ${}^s\mathbf{E}$, ${}^s\mathbf{H}^*$ and ${}^c\mathbf{E}$, ${}^c\mathbf{H}^*$, we arrive at

$$\oint_S [{}^{s(c)}\mathbf{E} \times {}^{c(s)}\mathbf{H}^*]\mathbf{n}dS = i\omega \int_V ({}^{s(c)}\mathbf{E}\varepsilon^{*c(s)}\mathbf{E}^* - {}^{c(s)}\mathbf{H}^*\mu^{s(c)}\mathbf{H})dV. \tag{42}$$

Here $\mathbf{n}$ is the outward normal to the surface.

The relations (42) may be named as the second Lorentz lemma for complex power in the case of source-free volume.

We shall let $V_1 = V_2 \to 0$ in such a way as to keep the aperture Ω inside the volume down to the limit $V = 0$. Under the "condition at an edge," V is a source-free area, and therefore the right-hand part of (42) must vanish. In the limit, the surface integral along

Ω is traversed twice with the normal $\mathbf{n}$ oppositely directed. Considering the continuity of the fields at the aperture, we get

$$\int_{\Omega}[{}^{s(c)}\mathbf{E}_1 \times {}^{c(s)}\mathbf{H}_1^*]\mathbf{n}dS = \int_{\Omega}[{}^{s(c)}\mathbf{E}_2 \times {}^{c(s)}\mathbf{H}_2^*]\mathbf{n}dS. \tag{43}$$

Going to the tangential field components, we can rewrite (43) in the equivalent matrix form

$${}^{s(c)}\mathbf{b}\left({}^{s(c)}\mathbf{u}_1, \frac{\partial^{c(s)}\mathbf{u}_1^{\dagger}}{\partial \boldsymbol{n}}\right){}^{c(s)}\mathbf{b}^{\dagger} = {}^{s(c)}\mathbf{b}\left({}^{s(c)}\mathbf{u}_2, \frac{\partial^{c(s)}\mathbf{u}_2^{\dagger}}{\partial \boldsymbol{n}}\right){}^{c(s)}\mathbf{b}^{\dagger}.$$

Since ${}^{s(c)}\mathbf{b} \neq \mathbf{0}$ are arbitrary, we obtain the sought-for relation (16).

Appendix B

Below it is demonstrated that the operator $\mathbf{DD}^T$ is bounded, whereas the operator $\mathbf{B}$ is compact in the Hilbert space ℓ_2. The latter is split into a small-in-norm operator and a remainder.

Let $\boldsymbol{\varphi}(x) = \{\phi_m(x)\}$ and $\boldsymbol{\upsilon}(x) = \{\upsilon_m(x)\}$ be orthonormal systems of eigenfunctions of two different Sturm-Liouville problems, which satisfy the same homogeneous boundary conditions at the ends of the interval $x \in \Omega$. Taking into account the properties of Fourier's coefficients of the function from the Hölder space (see, e.g., Tolstov, 1976), we find that the matrix operator

$$(\boldsymbol{\varphi}, \boldsymbol{\upsilon}^T) : \begin{cases} h_\gamma \to h_\gamma \\ c_0 \to c_0 \end{cases}$$

is continuous. Here $c_0 \subset \ell_\infty$ is the space of the bounded sequences that converge to zero. Again, it can be easily checked that the diagonal operator

$$\mathbf{I}_\beta^{\pm 1/2} : \begin{cases} h_\gamma \rightleftarrows \ell_2 \\ \ell_2 \rightleftarrows c_0 \end{cases}$$

is bounded when $\beta_m = O(m)$. Thus, the operators

$$\mathbf{F}_{\beta,\beta} = \mathbf{I}_\beta^{1/2}(\boldsymbol{\varphi}, \boldsymbol{\upsilon}^T)\mathbf{I}_\beta^{-1/2}, \quad \mathbf{F}_{\beta,\beta}^T = \mathbf{I}_\beta^{-1/2}(\boldsymbol{\upsilon}, \boldsymbol{\varphi}^T)\mathbf{I}_\beta^{1/2} \tag{44}$$

are bounded $\ell_2 \to \ell_2$ as the products of the above operators. Since $\gamma_m, \nu_m = O(m)$, $m \gg 1$, from (44) it follows that $\mathbf{D}, \mathbf{D}^T$, and hence $\mathbf{DD}^T$, are bounded.

Next, the matrix operator $\mathbf{B} = \mathbf{DD}^T - \mathbf{I}$ in the form

$$\mathbf{B} = \mathbf{I}_\gamma^{1/2}(\boldsymbol{\varphi}, K\boldsymbol{\varphi}^T)\mathbf{I}_\gamma^{1/2} \tag{45}$$

is expressed in terms of the compact integral operator

$$K\varphi_n(x) = \int_{\Omega} K(x, x')\varphi_n(x')dx' \tag{46}$$

with the kernel $K(x, x') = G_2(r, 0 \mid r', 0) - G_1(x, 0 \mid x', 0)$. Here

$$G_j = \begin{cases} \sum_{m=(0)1}^{\infty} \frac{1}{\gamma_m} \varphi_m(x)\varphi_m(x'), & j = 1, \\ \sum_{n=(0)1}^{\infty} \frac{1}{\nu_n} \psi_n(r)\psi_n(r'), & j = 2, \end{cases} \tag{47}$$

is the projection of Green's function of the corresponding region onto the junction plane. Clearly, $K(x, x') \in C^0(\Omega \times \Omega)$ is a symmetrical function with respect to interchanging of the variables. Indeed, extracting the static part, we get

$$K(x, x') = \frac{1}{\pi}\left[\ln\left|\sin\left(\frac{\pi(x - x')}{4a}\right) \Big/ \sin\left(\frac{\pi}{2w}\ln\left(\frac{r'}{r}\right)\right)\right| \right.$$
$$\left. \mp \ln\left|\sin\left(\frac{\pi(2a - x - x')}{4a}\right) \Big/ \sin\left(\frac{\pi}{2w}\ln\left(\frac{rr'}{r_2^2}\right)\right)\right|\right] + \Phi(x, x')\begin{pmatrix} H \\ E \end{pmatrix}, \tag{48}$$

where $w = \ln(r_2/r_1)$ and the function Φ is differentiable an infinite number of times. Hence, K is the trace class (nuclear) operator (see, e.g., *Encyclopaedia of Mathematics*, 1995).

Now let $\boldsymbol{\upsilon}(x) = \{\upsilon_m(x)\}$ be the complete orthonormal system of eigenfunctions of the operator $(K^\dagger K)^{1/2}$, which correspond to the positive eigenvalues $\{\alpha_m\}$. Then $K\boldsymbol{\varphi}(x) = \boldsymbol{\upsilon}^T(x)\mathbf{I}_\alpha(\boldsymbol{\upsilon}, \boldsymbol{\varphi}^T)$, where $\mathbf{I}_\alpha$ is a nuclear operator (Richtmyer, 1978). Substituting this representation to the formula (45) and using (44), we arrive at

$$\mathbf{B} = \mathbf{I}_\gamma^{1/2}(\boldsymbol{\varphi}, \boldsymbol{\upsilon}^T)\mathbf{I}_\alpha(\boldsymbol{\upsilon}, \boldsymbol{\varphi}^T)\mathbf{I}_\gamma^{1/2} = [\mathbf{F}_{\gamma,\nu}\mathbf{I}_\nu^{1/2}]\mathbf{I}_\alpha[\mathbf{I}_\nu^{1/2}\mathbf{F}_{\gamma,\nu}^T].$$

Since the diagonal operator $\mathbf{I}_\nu^{1/2}\mathbf{I}_\alpha\mathbf{I}_\nu^{1/2} = \mathbf{I}_{\nu\alpha}$ is compact, $\mathbf{B} : \ell_2 \to \ell_2$ is compact too.

Separating N terms in the series (47), let us represent the kernel of the nuclear operator (46) as a sum $K(x, x') = K'(x, x') + \tilde{K}(x, x')$. Here the first summand is the difference of two degenerated subkernels and the second summand is the remainder. Then corresponding integral operator $\tilde{K}$ is a small-in-norm one when $N \gg 1$. Hence, the norm of the compact operator $\tilde{\mathbf{B}} = \mathbf{I}_\gamma^{1/2}(\boldsymbol{\varphi}, \tilde{K}\boldsymbol{\varphi}^T)\mathbf{I}_\gamma^{1/2}$ vanishes when $N \to \infty$.

Appendix C

Since $\mathbf{R}$ is a compact operator, its entire spectrum consists of point spectrum σ_p ($\mathbf{R}$) and the point zero that belongs to the continuous or residual spectrum (see, e.g., Richtmyer, 1978). Below it is shown that this spectrum lies in the interior of unit disk.

Let $\mathbf{b}$ be a normalized eigenvector of $\mathbf{R}$ (i.e., $\|\mathbf{b}\| = 1$). Then, according to (20) we have

$$\mathbf{b}(\mathbf{I} + \mathbf{R})\mathbf{U}_P^2(\mathbf{I} - \mathbf{R}^*)\mathbf{b}^\dagger = \mathbf{b}\mathbf{T}\mathbf{U}_Q^2\mathbf{T}^\dagger\mathbf{b}^\dagger.$$

Taking into account that $\mathbf{bT} = \mathbf{t}$ and $\mathbf{bR} = \lambda\mathbf{b}$, we obtain the equality

$$(1 + \lambda)(1 - \lambda^*)\mathbf{b}\mathbf{U}_P^2\mathbf{b}^\dagger = \mathbf{t}\mathbf{U}_Q^2\mathbf{t}^\dagger.$$

Its equivalent form is

$$(1+\lambda)(1-\lambda^*) = s, \tag{49}$$

where

$$\begin{gathered} s = \frac{\mathbf{t}\mathbf{U}_Q^2\mathbf{t}^\dagger}{\mathbf{b}\mathbf{U}_P^2\mathbf{b}^\dagger} = \chi^2[(1-\tau_1^2)(1-\tau_2^2) + \tau_1^2\tau_2^2 + i(\tau_1^2-\tau_2^2)][(1-\tau_1^2)^2+\tau_1^4]^{-1}, \\ \chi = \|\mathbf{t}\| \neq 0, \quad \tau_1^2 = \sum_{m=(0)1}^{P} |b_m|^2, \quad \tau_2^2 = \chi^{-2} \sum_{n=(0)1}^{Q} |t_n|^2. \end{gathered} \tag{50}$$

Considering the evident properties $0 \le \tau_j < 1$, $j = 1, 2$, from (50) we find that $\mathrm{Re}(s) > 0$. However, according to (49) $\mathrm{Re}(s) = 1 - |\lambda|^2$, and hence we arrive at $|\lambda| < 1$.

The condition $\chi \neq 0$ has the clear electrodynamic sense; namely, there is no realizable source of modes (propagating or evanescent) such that after the junction the transmitted field is absent.

Note that the localization of the point spectrum of the generalized scattering matrix was studied by Mittra and Lee (1971). In the general case, the complete analysis of the entire spectrum of a reflection operator was considered in Shestopalov, Kirilenko, and Masalov (1984).